

**Math 320 (Winter 2006)**  
**Introduction to Real Analysis II**  
**Assignment 5 Solutions**

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**Q1.** Clearly,  $f(E) \subseteq \overline{f(E)}$ , and so  $E \subseteq f^{-1}(\overline{f(E)})$ . Since  $f$  is continuous,  $f^{-1}(\overline{f(E)})$  is closed. But by definition,  $\overline{E}$  is the smallest closed set containing  $E$ . Therefore,  $\overline{E} \subseteq f^{-1}(\overline{f(E)})$ .

By considering the function  $f : [1, \infty) \rightarrow \mathbb{R}$  by  $f(x) = 1/x$ . Also, let  $E = [1, \infty)$ , and so  $E$  is closed and we have

$$f(E) = f(\overline{E}) = f([1, \infty)) = (0, 1],$$

which is a proper subset of  $[0, 1]$ .

**Q2.** Note that  $Z(f) = f^{-1}(\{0\})$ . Therefore,  $Z(f)$  is closed because  $\{0\}$  is also closed.

**Q3.** We know that  $\overline{E} = X$ . For all  $p \in X$ , there exists a sequence  $\{p_n\} \subseteq E$  such that  $p_n \rightarrow p$ .

Since  $f$  and  $g$  are continuous, then  $f(p_n) \rightarrow f(p)$  and  $g(p_n) \rightarrow g(p)$ . Also, since  $p_n \in E$ , we have  $f(p_n) = g(p_n)$ . Therefore,  $g(p_n) \rightarrow f(p)$  and thus  $f(p) = g(p)$ .

**Q4.** Let  $\pi_X : X \times Y \rightarrow X$  and  $\pi_Y : X \times Y \rightarrow Y$  be projections on  $X$  and  $Y$  respectively. Take

$$\pi_X(x, y) = x, \quad \pi_Y(x, y) = y.$$

Observe that both  $\pi_X$  and  $\pi_Y$  are continuous.

Suppose  $f$  is continuous. Let  $\{U_\alpha\}$  be a collection of open sets in  $X \times Y$  which covers  $G$ . Note that  $\pi_X(U_\alpha)$  is open in  $X$  and  $\{\pi_X(U_\alpha)\}$  forms an open cover for  $E$ .

Since  $E$  is compact, there would be a finite subcover  $\{\pi_X(U_{\alpha_i})\}_{i=1}^n$  covering  $E$ . The corresponding finite subcover  $\{U_{\alpha_i}\}_{i=1}^n$  covers  $G$ , and so  $G$  is compact.

Conversely, suppose  $G$  is compact. Let  $V$  be a closed set in  $f(E)$ . Then  $\pi_X^{-1}(V) \cap G$  is closed in  $G$ , and so its also compact in  $G$ . Since  $\pi_X$  is

continuous, then  $\pi_X(\pi_Y^{-1}(V) \cap G)$  is also compact. But

$$\pi_X(\pi_Y^{-1}(V) \cap G) = f^{-1}(V),$$

and so  $f$  is continuous.

**Q5.** Suppose  $f$  is an open continuous map but not a monotonic increasing (the argument is the same for decreasing), then there are points  $x_1 < x < x_2$  such that  $f(x) > \max\{f(x_1), f(x_2)\}$ . Since  $f$  is continuous, then

$$\max\{f(x) : x \in [x_1, x_2]\} = f(x_0) > \max\{f(x_1), f(x_2)\},$$

for some  $x_0 \in (x_1, x_2)$ . Therefore,  $f((x_1, x_2))$  is either  $(f(x_1), f(x_0)]$  or  $(f(x_2), f(x_0)]$ , neither of which is open. This contradicts the fact that  $f$  is an open map.

**Q6.** Claim that for  $a < x < z < b$ ,

$$\frac{f(z) - f(x)}{z - x} \leq \frac{f(b) - f(x)}{b - z}.$$

Let  $z = \lambda x + (1 - \lambda)b$ . ie.,

$$1 > \lambda = \frac{z - b}{x - b} = \frac{b - z}{b - x} > 0.$$

Since  $f$  is convex,

$$\begin{aligned} f(z) &\leq \lambda f(x) + (1 - \lambda)f(b) \\ &= \left(\frac{b - z}{b - x}\right) (f(x) - f(b)) + f(b). \end{aligned}$$

$$\begin{aligned} f(z) - f(x) &\leq \left(\frac{b - z}{b - x}\right) (f(x) - f(b)) + f(b) - f(x) \\ &= \left(\frac{b - z - b + x}{b - x}\right) f(x) + \frac{b - x - b + z}{b - x} f(b) \\ &= \frac{x - z}{b - x} f(x) + \frac{z - x}{b - x} f(b). \end{aligned}$$

$$\begin{aligned} \frac{f(z) - f(x)}{z - x} &\leq -\frac{f(x)}{b - x} + \frac{1}{b - x} f(b) \\ &= \frac{f(b) - f(x)}{b - x}. \end{aligned}$$

This proves the last inequalities.

We need to prove  $f$  is continuous at  $x \in (a, b)$ . If  $f(b) = f(x)$  or  $f(x) = f(a)$  is trivial. For all  $\epsilon > 0$ , let

$$\delta = \frac{\epsilon}{\max \left\{ \frac{f(b)-f(x)}{b-x}, \frac{f(x)-f(a)}{x-a} \right\}}.$$

If  $|z - x| < \delta$ , then

$$\begin{aligned} \frac{f(z) - f(x)}{z - x} &\leq \frac{f(b) - f(x)}{b - x} \\ \Rightarrow |f(z) - f(x)| &\leq |z - x| \cdot \max \left\{ \frac{f(b) - f(x)}{b - x}, \frac{f(x) - f(a)}{x - a} \right\} \\ &< \epsilon. \end{aligned}$$

Therefore,  $f$  is continuous at  $x$ .

If  $f$  is increasing and convex, then we know that

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

So

$$\begin{aligned} &f(g(\lambda x + (1 - \lambda)y)) \\ &\leq f(\lambda g(x) + (1 - \lambda)g(y)) && \text{(since } g \text{ convex, } f \text{ increasing)} \\ &\leq \lambda f(g(x)) + (1 - \lambda)f(g(y)). && \text{(since } f \text{ convex)} \end{aligned}$$

Therefore,  $f \circ g$  is convex.