Homework 10
Math 425
Real Analysis

Qn.1 Let ν be a signed measure on $(X, \mathcal{M})$.
(a) $L^1(\nu) = L^1(|\nu|)$.
(b) If $f \in L^1(\nu)$, $\left|\int f d\nu\right| \leq \int |f| d|\nu|$.
(c) If $E \in \mathcal{M}, |\nu|(E) = \sup\{|\int_E f d\nu| : |f| \leq 1\}$.

Qn.2 If ν is a signed measure and λ, μ are positive measures such that $\nu = \lambda - \mu$, then $\lambda \geq \nu^+$ and $\mu \geq \nu^-$.

Qn.3 If ν_1, ν_2 are signed measures that both omit the value $+\infty$ or $-\infty$, then $|\nu_1 + \nu_2| \leq |\nu_1| + |\nu_2|$.

Qn.4 Show that $\nu \ll \mu$ iff $|\nu| \ll \mu$ iff $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

Qn.5 For $j = 1, 2$, let μ_j, ν_j be $\sigma-$finite measures on $(X_j, \mathcal{M}_j)$ such that $\nu_j \ll \mu_j$. Then $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1)\frac{d\nu_2}{d\mu_2}(x_2).$$

Qn6. Let $X = [0, 1], \mathcal{M} = \mathcal{B}_{[0,1]}, m =$Lebesgue measure, and $\mu =$counting measure on $\mathcal{M}$. Show that
(a) $m \ll \mu$ but $dm \neq f d\mu$ for any f.
(b) μ has no Lebesgue decomposition with respect to m.

Qn.7 If ν, μ are complex measures and λ is a positive measure, then $\nu \perp \mu$ iff $|\nu| \perp |\mu|$ and $\nu \ll \lambda$ iff $|\nu| \ll \lambda$.

Qn.8 A useful variant of the Hardy-Littlewood maximal function is

$$H^* f(x) = \sup\left\{\frac{1}{m(B)}\int_B |f(y)| dy : B \text{ is a ball and } x \in B\right\}.$$

Show that $Hf \leq H^* f \leq 2^n Hf$.

Qn.9 If F is increasing on $\mathbb{R}$, show that $F(b) - F(a) \geq \int_a^b F'(t) dt$.