Midterm (Solution)

Math 425

Real Analysis

Qn.1 A T_1 space X is countably compact if and only if every infinite subset of X has an accumulation point.

Solution: Suppose A is an infinite set in X. Then for any $y \in A, x \in X \backslash A$, there is open set U_y such that $x \notin U_y$ and $y \in U_y$. Since $\cap_{y \in A} \overline{U_y} \neq \phi$. Then choose $x_0 \in \cap_{y \in A} \overline{U_y}$. Suppose U is a nbd of x_0, then $U \cap U_y \neq \phi$ for all y. Thus $A \cap (U\{x_0\}) \neq \phi$. Therefore x_0 is an accumulation point of A.
Suppose every finite subset of X has an accumulation point. Then for each sequence $\{x_n\}$, $\{x_n\}$ has an accumulation point x. Therefore, we can find a subset $\{x_{n_k}\}$ such that $\{x_{n_k}\}$ converges to x. Then by Qn 2 of Homework 3, X is countably compact.

Qn.2 Give **four** separations axioms.
In $\mathbb{R}^n$ let $\mathcal{B}$ be the family of sets $\{x : p(x) \neq 0\}$, where p is a polynomial in n variables. Let $\mathcal{T}$ be the family of all finite intersections $O = B_1 \cap B_2 \cap \cdots \cap B_k$ from $\mathcal{B}$. Show that $\mathcal{T}$ gives a topology for $\mathbb{R}^n$ which is T_1 and compact but not T_2.

Solution: Since polynomials are continuous functions, so $\{x : p(x) \neq 0\} = p^{-1}(\mathbb{R} \backslash \{0\})$ is open sets in $\mathbb{R}^n$ and $\mathcal{B}$ is a collection of open sets. Clearly, for any $x \in \mathbb{R}^n$, $x \in \{x : p(x) \neq 0\} \in \mathcal{B}$ for constant polynomial p.
For any $O_1, O_2 \in \mathcal{B}$, let $O_1 := \{x : p_1(x) \neq 0\}$ and $O_2 := \{x : p_2(x) \neq 0\}$. Then $O_1 \cap O_2 = \{x : p_1(x)p_2(x) \neq 0\} \in \mathcal{B}$ Therefore, by Proposition 4.3, $\mathcal{B}$ forms a base for a topology. By Proposition 4.4, $\mathcal{T}$ is a topology generated by $\mathcal{B}$.
Note that $\mathcal{T}$ satisfies the finite intersection property. So $\mathcal{T}$ is compact.
Since $\{x_0\} := \mathbb{R}^n \backslash \{x \in \mathbb{R}^n : x_j - x_0 \neq 0\}$, so $\{x_0\}$ is closed for any $x_0 \in \mathbb{R}^n$. Hence $\mathcal{T}$ is T_1 by Proposition 4.7.
Since any two open sets has nonempty subsets, so $\mathcal{T}$ is not Hausdorff.

Qn.3 Let X be a set. Give definitions of σ-algebra, $\mathcal{M}$, and measure, μ, of X. Prove that if $\{E_j\}_{j=1}^{\infty} \subseteq \mathcal{M}$ and

$$E_1 \supset E_2 \supset E_3 \supset \cdots ,$$

and $\mu(E_1) < \infty$, then

$$\mu\left(\bigcap_{j=1}^{\infty} E_j\right) = \lim_{n\to\infty} \mu(E_j).$$

Give an example to show that the condition $\mu(E_1) < \infty$ is essential.

Solution: Refer to the Theorem 1.8 of the textbook. An example is: let μ be counting measure on $(\mathbb{N}, \mathbf{P}(\mathbb{N}))$ and let $E_j = \{n : n \geq j\}$; then $\cap_{j=1}^{\infty} E_j = \phi$.