Supplement to the paper
"Continued Fractions of Tails of Hypergeometric Series"
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by
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The above-mentioned paper contains, among others, the following formula

$$\frac{\pi}{2} - 2\sum_{k=1}^{N/2} \frac{(-1)^{k-1}}{2k-1} \sim \sum_{m=0}^{\infty} \frac{E_{2m}}{N^{2m+1}} = \frac{1}{N} - \frac{1}{N^3} + \frac{5}{N^5} - \frac{61}{N^7} + \ldots, \tag{2}$$

where N is positive integer divisible by 4, while the numerators $1, -1, 5, -61, \ldots$ are the Euler numbers $E_0, E_2, E_4, E_6, \ldots$. It may be interesting to know that there is also a formula which "can do" without the Euler numbers, namely

$$\frac{\pi}{2} - 2\sum_{k=1}^{N/2} \frac{(-1)^{k-1}}{2k-1} = \frac{1}{N + F_N}. \tag{2a}$$

Here F_N denotes the beautiful continued fraction

$$\frac{1^2}{N} + \frac{2^2}{N} + \frac{3^2}{N} + \frac{4^2}{N} + \frac{5^2}{N} + \cdots.$$

Note that the left-hand sides of (2) and (2a) are identical. Only the right-hand sides differ from each other, because (2) uses the Euler numbers, which are not needed in (2a).

The equation (2a) can be derived from other formulas which, until now, are published and proved only in German, namely in the journal article: W. Pigulla, "Konvergenzbeschleunigung mit Hilfe von Kettenbrüchen", Elemente der Mathematik 59 (2004), 58-64.

It we test (2a) with $N = 4$, we obtain

$$\frac{\pi}{2} - 2\left(1 - \frac{1}{3}\right) = \cfrac{1}{4 + \frac{1}{4} + \frac{4}{4} + \frac{9}{4} + \frac{16}{4} + \frac{25}{4} + \ldots} = 0.2374\cdots.$$

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