Sparse Polynomials

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Let $f \in R[x_1,\dots,x_n]$, R a ring.

Let $f = \sum_{i=1}^{t} a_i \cdot M_i(x_1,\dots,x_n)$ where $a_i \in R$, $a_i \neq 0$.

M_i: monomials.

If $\deg(f, x_i) = d_i$ then f may have upto $t = \prod_{i=1}^{n}(d_i+1)$ terms.

E.g. $n=2$, $d_1=3$, $d_2=2$. $t \le (3+1)(2+1) = 12$.

deg x_1 \ deg x_2	0	1	2
0	1	x_2	x_2^2
1	x_1	x_1x_2	$x_1x_2^2$
2	x_1^2	$x_1^2 \cdot x_2$	$x_1^2 \cdot x_2^2$
3	x_1^3	$x_1^3 \cdot x_2$	$x_1^3 \cdot x_2^2$

We say f is sparse if

$t = \#f \ll \prod_{i=1}^{n}(d_i+1)$.

? $t \le \sqrt{\prod_{i=1}^{n}(d_i+1)}$.

E.g. $f = 3x_1 + 2x_1x_2 + 5x_2x_1^2$

Alternatively if $\deg(f) = d$ then f may have upto $\binom{n+d}{d}$ terms.

E.g. $d=3$, $n=2$ $\binom{n+d}{d} = \binom{2+3}{3} = \frac{5!}{2!\,3!} = \frac{5 \cdot 4}{2!} = 10$ terms.

1	x_1	x_1^2	x_1^3
x_2	$x_1 \cdot x_2$	$x_1^2 \cdot x_2$	
x_2^2	$x_1x_2^2$		
x_2^3			

We say f is sparse if

$t \ll \binom{n+d}{d}$.

I use $t \le \sqrt{\binom{n+d}{d}} = \sqrt{10}$

So $f = 1 - x^3 - y^3$ is sparse.

$$T_3 = \begin{bmatrix} u & v & w \\ v & u & v \\ w & v & u \end{bmatrix}$$

$\det(T_3) = u^3 - 2uv^2 - uw^2 + 2v^2w$. $t = 4$. sparse.

$\binom{n+d}{d} = \binom{6}{3} = 20$.

In ctile most polynomials with $n \ge 3$ are sparse.

Sparse Polynomial Interpolation.

Let $f \in k[x_1, \ldots, x_n]$ where k is a field.

Suppose f is given by a black-box $B : k^n \to k$.

Suppose $f = \sum_{i=1}^{t} a_i M_i(x_1, \ldots, x_n)$.

How many points in k^n do we need to interpolate f?

I.e. to recover the $a_i \in k$ and the monomials M_i.

Let $d_i = \deg(f, x_i)$.

Since $\#f \le \prod_{i=1}^{n} (d_i + 1) = D$ we may need D values of f.

E.g. $f(x_1, x_2) = \sum_{j=0}^{d_1} \left(\sum_{i=0}^{d_2} a_{ij} x_2^i \right) \cdot x_1^j$

Pick $\beta_0, \beta_1, \ldots, \beta_{d_2} \in k$.

Interpolate

$$f(x_1, \beta_0) = \cdot + \cdot\, x_1 + \cdots + \cdot\, x_1^{d_1} \quad \text{using } d_1+1 \text{ points for } x_1$$

$$f(x_1, \beta_1) = \cdot + \cdot\, x_1 + \cdots + \cdot\, x_1^{d_1} \quad \text{using } d_1+1 \text{ points for } x_1$$

$$\vdots$$

$$f(x_1, \beta_{d_2}) = \cdot + \cdot\, x_1 + \cdots + \cdot\, x_1^{d_1} \quad \text{using } d_1+1 \text{ points.}$$

$$\Longrightarrow f(x_1, x_2) = f_0(x_2) + f_1(x_2) \cdot x_1 + \cdots + f_{d_1}(x_2) \cdot x_1^{d_1} \qquad \text{So } (d_2+1)(d_1+1) \text{ points}$$

Suppose f is sparse. Can we improve on this?

	# values of f	$f = 1 \cdot x_1^d + 1 \cdot x_2^d + \cdots + 1 \cdot x_n^d$
Dense method.	$\prod_{i=1}^{n} (d_i + 1)$	$= (1+d)^n \leftarrow$ is exponential in n.
Zippel PhD 1979	$\le \left(t \sum_{i=1}^{n} d_i \right) + 1$	$= t n d + 1$
Ben-Or/Tiwari PhD 1988	$2T$ where $T \ge t$?	$2T$