

A Graph Theory Package for Maple, Part II: Graph Coloring, Graph Drawing, Support Tools, and Networks.

Mohammad Ali Ebrahimi, Mohammad Ghebleh, Mahdi Javadi
Michael Monagan and Allan Wittkopf
Department of Mathematics, Simon Fraser University,
8888 University Drive,
Burnaby, BC, Canada V5A 1S6.
{mebrahim,mghebleh,mmonagan,sjavadi,wittkopf}
@cecm.sfu.ca

1 Introduction

This is the second paper detailing a new Maple package for graph theory, called `GraphTheory`, [3], that is being developed at Simon Fraser University as part of our MITACS research project. The package is intended for both research and teaching. It supports directed and undirected graphs, weighted and unweighted graphs, and networks, but not multigraphs.

The package includes procedures for drawing graphs in 2 and 3 dimensions. To make the package easy to use we have added Maple style ‘context menus’. This provides access to many of the commands in the package via using menus and the mouse.

In this paper we describe new functionality added to support graph input, storing of arbitrary information about edges and vertices, importing graphs from and exporting graphs to other software, including support for MetaPost for use with \LaTeX .

We describe a graph drawing algorithm that models a graph as a physical system that leads to a system of second order differential equations that needs to be solved. This approach gives very good drawings of graphs. It is limited in that its complexity is inherently $O(n^2)$ where n is the number of vertices in the graph. Our implementation is effective for n up to about 100.

We also describe the facilities for graph coloring and for networks.

There are many texts on graph theory. The primary algorithms reference that we have used was *Introduction to Algorithms* by Cormen, Leiserson, Rivest and Stein. See [1]. Also [8], Gabriel Valiente’s *Algorithms on Trees and Graphs* which focusses on the graph isomorphism problem. Another good text devoted

to graph theory algorithms is [4], Dieter Jungnickel's *Graphs, Networks, and Algorithms*.

2 Defining Graphs

2.1 The Graph and Digraph commands

The commands `Graph` and `Digraph` are used to construct an undirected or directed graph, respectively. Here we describe the `Graph` command.

The following parameters of the graph can be given *in any order* as arguments of the `Graph` command: number of vertices, vertex labels, edges/arcs, the adjacency matrix, the weight matrix, the keywords `directed`, `undirected`, `weighted`, and `unweighted`.

Vertex labels are specified in a list (in a fixed order). Each label can be either an integer, a symbol, or a string.

Edges are of the form $\{x,y\}$ and arcs are of the form $[x,y]$ where x and y are valid vertex labels. A convenient solution for entering the edges of a graph is to use trails. For example `Trail(1,2,3,4,2,3,5)` for an undirected graph is short for the edges $\{\{1,2\},\{2,3\},\{3,4\},\{2,4\},\{3,5\}\}$.

If the graph to be constructed is weighted, then the weights of edges/arcs can be given in a weight matrix, or in the set of edges in the form $[e,w]$, where e is an edge/arc and w is the weight of e . If no weight is assigned to an edge, it is assigned the weight 1 by default. Here are some examples.

```
> G := Graph(5);
      Graph 1: an undirected unweighted graph with 5 vertices and 0 edge(s)

> Vertices(G), Edges(G);
      [1, 2, 3, 4, 5], {}

> Graph([a,b,c,d,e], {{a,b}, {b,c}, {c,e}, {a,c}});
      Graph 2: an undirected unweighted graph with 5 vertices and 4 edge(s)
> Graph({{a,b}, {b,c}, {c,e}, {a,c}});
      Graph 3: an undirected unweighted graph with 5 vertices and 4 edge(s)
> G := Graph([a,b,c,d,e], Trail(c,a,b,c,e));
      G := Graph 5: an undirected unweighted graph with 5 vertices and 4 edge(s)
> Edges(G);
      {{c, a}, {c, b}, {c, e}, {a, b}}
> W := Matrix(6, {(1,2)=2, (2,3)=4, (2,6)=7, (3,4)=6, (3,5)=5, (4,5)=3},
      shape=symmetric);
> G := Graph([1..6], W);

> Edges(G, weights);
      {{{1, 2}, 2}, {{2, 3}, 4}, {{2, 6}, 7}, {{3, 4}, 6}, {{3, 5}, 5}, {{4, 5}, 3}}
```

```
> A := Matrix([[0,1,1,1],[1,0,1,0],[1,1,0,1],[1,0,1,0]]);
> Edges( Graph([a,b,c,d], A) );
      {{a,b},{a,c},{a,d},{b,c},{c,d}}
```

A graph may also be defined by specifying its list of neighbors. This argument must be of type `list` or `Array` of sets of vertices. For example:

```
> A := Array([[2,3], {1,3,4,5}, {1,2,4,5}, {2,3}, {2,3}]);
> Edges( Graph([v1,v2,v3,v4,v5], A) );
      {{v1,v2},{v1,v3},{v2,v3},{v2,v4},{v2,v5},{v3,v4},{v3,v5}}
```

2.2 The data structure

The `GraphTheory` package stores graphs in Maple as functions. The following is the general form of a graph generated by the `GraphTheory` package:

```
GRAPHLN(dir, wt, vlist, listn, ginfo, ewts)
```

In this structure, `dir` and `wt` are symbols indicating whether the graph is `directed` or `undirected` and whether it is `weighted` or `unweighted`. The argument `vlist` is the list of vertex labels of the graph. Each label can be of type `integer`, `string` or `symbol`. Internally, the vertices are labelled by numbers $1, 2, \dots, n$ where n is the number of vertices of the graph. The adjacency structure of the graph is stored in `listn` which is an `Array` of sets. The i th entry in `listn` is the set of neighbors of the i th vertex. If the graph is weighted, the edge weights are stored in `ewts` which is of type `Matrix`. Additional information such as vertex positions and vertex and edge information will be stored in `ginfo` which is a of type `table`.

The choice of ‘list of neighbors’ as the graph data structure provides makes it efficient to insert and delete new edges in a graph but adding and removing vertices is more expensive – it requires making a copy of the graph structure. Another restriction of the list of neighbors as the graph structure is that multiple edges are not supported.

2.3 Attributes

Using attributes, the user may attach arbitrary information to the graph, its vertices, and edges. Each attribute is an equation of the form `tag = value`. We illustrate with examples.

```
> G := Graph( Trail(a, b, c, d, b) );
> SetVertexAttribute(G, b, msg="text");
> SetVertexAttribute(G, c, [msg="message", timesvisited=10]);
> GetVertexAttribute(G, c, timesvisited);
      10
> GetVertexAttribute(G, b, timesvisited);
      FAIL
> SetEdgeAttribute(G, {a,b}, msg="weak edge");
```

```

> SetEdgeAttribute(G, {c,d}, [tag=fake,visited=true,cost=3]);
> DiscardEdgeAttribute(G, {c,d}, tag);
> GetEdgeAttribute(G, {c,d});
    [visited = true, cost = 3]

```

2.4 Special graphs

The definitions of some special graphs are included in the `SpecialGraphs` submodule of the `GraphTheory` package. The following is a list of these special graphs.

ClebschGraph	GridGraph	PappusGraph
CompleteBinaryTree	GrinbergGraph	PathGraph
CompleteGraph	GrotzschGraph	PayleyGraph
CompleteKaryTree	HeawoodGraph	PetersenGraph
CycleGraph	HerschelGraph	PrismGraph
DesarguesGraph	HypercubeGraph	ShrikhandeGraph
DodecahedronGraph	IcosahedronGraph	SoccerBallGraph
DoubleStarSnark	KneserGraph	SzekeresSnark
DyckGraph	LCFGraph	TetrahedronGraph
FlowerSnark	LeviGraph	TorusGridGraph
FosterGraph	MobiusKantorGraph	WebGraph
GeneralizedPetersenGraph	OctahedronGraph	WheelGraph
GoldbergSnark	OddGraph	

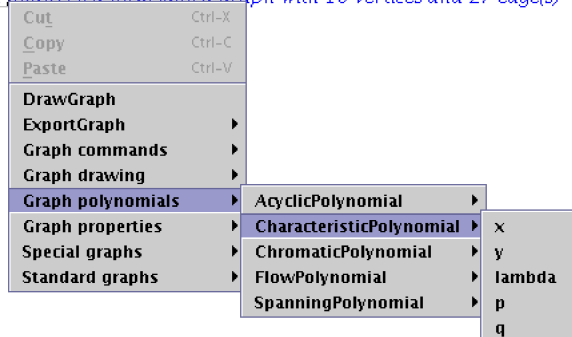
2.5 Context menu

The context menu is a convenient way to access some of the `GraphTheory` package commands. The context menu of a graph can be accessed in the Maple graphical user interface by right-clicking on the graph. An example is depicted in the following figure.

```

> RandomGraph(10, .7);
Graph 1: an undirected unweighted graph with 10 vertices and 27 edge(s)

```



3 Drawing Graphs

The `GraphTheory` package provides a command `DrawGraph` for drawing graphs and digraphs. If a graph doesn't contain drawing information, it is checked for being a tree or a bipartite graph in which cases it is drawn with respect to those structures. Otherwise the vertices are positioned on a circle. To force a style of drawing, one may specify a second argument which is of the form `style=...`. In the remainder of this section we describe one of these styles, namely the `spring` style. The spring method may be used to generate 2-dimensional or 3-dimensional drawings of a graph. The default dimension is 2 and an extra argument `dimension=3` can be used to generate a 3D drawing. Once a drawing is generated it will be stored in the memory for future reference. To force generation of a new drawing, which is useful for the spring method which uses random initial positions, one may use the option `redraw` in the `DrawGraph` command.

Let G be a connected graph with n vertices V and m edges E . We want our algorithm to find an aesthetic layout of the graph that clearly conveys its structure and assign a location for each vertex and a route for each edge, so that the resulting drawing is 'nice'. The method we use models a graph as a *physical* system that leads to a system of second order differential equations that must be numerically solved.

We model edges in the graph with springs with zero rest length. We model vertices as nonphysical objects that exert a constant repulsive force on each of the other vertices. And we add damping to the physical system to dampen oscillations. Note that different variants of the differential equation spring model have been used, including springs with nonzero rest lengths [2], [5] modification of this method by an additional effect, temperature [6] as well as methods that dynamically change the model parameters as the system is integrated [7].

Thus we construct a system of second order differential equations where the equation for vertex i is the following:

$$\frac{d^2}{dt^2} X_i(t) = \sum_{j \in N(i)} a_{ij}(X_i(t) - X_j(t)) - b_i \frac{d}{dt} X_i(t) + \sum_{j \in V} c_{ij} \frac{X_i(t) - X_j(t)}{|X_i(t) - X_j(t)|}$$

where V is the set of vertices of G , $N(i)$ are the vertices adjacent to vertex i , X_i is the position of vertex i , a vector, and a_{ij}, b_i, c_{ij} are constants that must be defined for the problem. Thus if there are n vertices in the graph and we model the system in 2 dimensions, there are $2n$ second order differential equations in $2n$ unknowns. For the initial values, we set initial velocities to 0 and choose initial positions at random inside the unit square (in 2 dimensions) and unit box (in 3 dimensions).

This ODE system must then be integrated until it reaches a steady state. This is accomplished by use of an externally implemented fourth-order Runge-Kutta method with the graph ODE system built-in.

It is informative to note that the original model in use was a more physical model, in that the repulsive force obeyed an inverse square law. Alas, we

found that use of this approach caused the computation to *lock-up* before an aesthetically pleasing graph was obtained (nodes could not cross one-another), suffered from rather severe issues with respect to asymptotic behavior (the equilibrium was slow to reach), and sometimes resulted in a sub-optimal steady state (e.g. the natural description of a wheel graph with many vertices is an unstable equilibrium with respect to the inverse square law).

Finally, since a constant force model is in use, it is possible that the steady state can have overlapping vertices. To alleviate this problem, a post-processing integration is performed that steadily increases the repulsive force between overlapping vertices until they no longer overlap.

While graph drawing is a complicated problem in general, an advantage of this approach is that it requires no special knowledge about the structure of the graph such as whether it has articulation points or not. It also gives very good drawings of graphs. It is limited in that the complexity is at least $O(n^2)$. This is because the model of vertices being charged particles repelling each other means each differential equation has n non-zero terms in it. Our implementation can treat graphs with n up to 100 in a few seconds.

Damping Constants

In order to reconstruct good graphs from the randomly positioned initial vertices it is important that the spring model is neither over damped nor under damped. Since we are solving the system of differential equations numerically, reaching the steady state takes significantly more effort if the damping is not optimal. To avoid this difficulty, we choose the damping constants (b_i), spring constants (a_{ij}), and repulsion constants (c_{ij}) carefully. After experimenting with different graphs we have concluded that the constants must depend upon characteristics of the graph.

After some analysis, reasonable approximations for the optimal constants have been obtained, and the approach used is briefly outlined here.

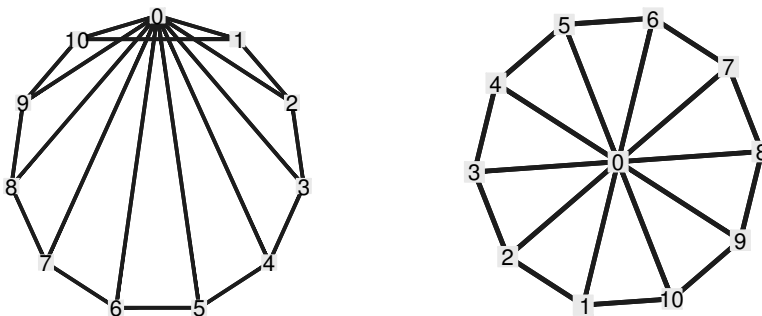
We have three sets of constants, and want to control three aspects of the problem: the time scale on which the steady state is reached, the length scale at the steady state, and the damping. With this in mind, we found the following relationships for the constants

$$a_{ij} = \frac{ae}{e_i e_j n}, \quad b_i = b\sqrt{a}, \quad c_{ij} = \frac{cae}{p_d(n^{\frac{1}{d}})},$$

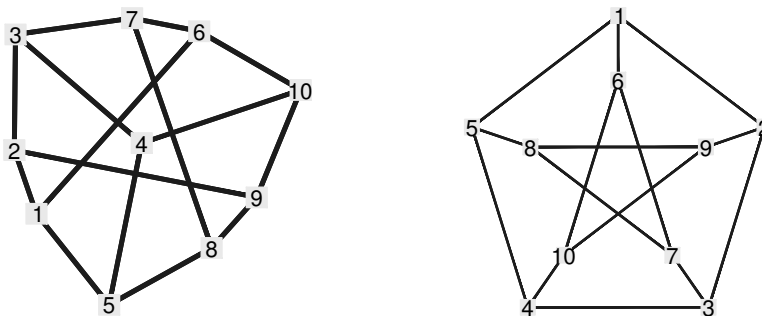
where e is the total number of edges in the graph, e_i is the number of edges connected to vertex i , n is the total number of vertices in the graph, d is the number of dimensions the graph is being drawn in, $p_d(x)$ is a degree $d + 2$ polynomial in x , and a, b, c are constants that control the time scale, length scale, and damping for the problem. Note that the $p_d(x)$ polynomial simply describes the repulsive force of n uniformly distributed vertices in a 1×1 square (for 2-d) or a $1 \times 1 \times 1$ cube (for 3-d) as measured at the center of any of the boundaries of the square/cube.

At this point, values for a, b, c were experimentally determined for the two and three dimensional cases, and it was found that fixed values for these provide very reasonable performance for a wide variety of graphs.

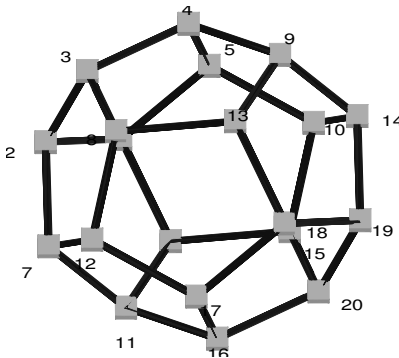
The first example shows that the algorithm recovers the ‘wheel graph’. The graph on the left was constructed by placing all vertices in a circle using the command `DrawGraph(G, style=circle)`. The graph on the right is using the algorithm. The command is `DrawGraph(G, style=spring)`.



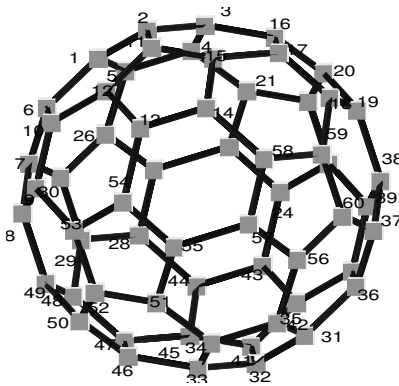
In this second example, we have constructed the Petersen graph. The graph on the right below is the default output of the `DrawGraph` command. This is usual way of drawing the Petersen graph. To obtain this output, the Graph-Theory package stores the vertex positions for this special graph. The graph on the left is the output of the spring algorithm. It is another symmetric drawing of the Petersen graph.



The final two examples are in three dimensions. The spring algorithm is easy to generalize to higher dimensions. In three dimensions, it is possible that the orientation of the result, of say the wheel graph, would end up showing as a line. To prevent this from happening, we rotate the graph through various angles and choose the orientation which maximizes the area of the convex hull of the vertices when projected onto the viewing plane. The first example is a dodecahedron. This graph has 20 vertices and 30 edges.



The second example is a graph of a soccer ball. The surface of a soccer ball consists of pentagons and hexagons. There are 60 vertices and 90 edges. The spring algorithm recovered the soccer ball in 0.546 seconds on an AMD 64 bit Opteron running at 2.2 GHz.



4 Graph Coloring

There are several commands implemented in the `GraphTheory` package for dealing with graph coloring problems. Since coloring problems are NP-Complete in general, the commands usually use brute-force search methods.

The two commands `ChromaticNumber` and `ChromaticIndex` return the chromatic number and the chromatic index (edge chromatic number) of a graph respectively. These commands also return an optimal coloring. To speed up the search, these commands calculate upper and lower bounds on the chromatic number of the given graph. The upper bound is obtained from a greedy coloring of the graph while the lower bound is the maximum of $\omega(G)$ and $\frac{n}{\alpha(G)}$ for a graph G where n is the number of vertices, $\omega(G)$ is the clique number, and $\alpha(G)$ is the independence number of G . The call `ChromaticNumber(G, bound)` calculates and returns the bounds without searching for the chromatic number. The commands `IsKColorable` and `IsKEdgeColorable` can be used to check if a graph is k -colorable or edge k -colorable for a fixed number k .

Circular colorings of graphs are also supported in the `GraphTheory` package. The commands `CircularChromaticNumber` and `CircularChromaticIndex` return the circular chromatic number and the circular edge chromatic number of a given graph along with the corresponding optimal circular colorings. The commands `IsKDColorable` and `IsKDEdgeColorable` are used to check (k, d) -colorability or edge (k, d) -colorability for a fixed pair (k, d) of integers.

From König's theorem we know that the chromatic index of every bipartite graph is equal to the maximum degree of a vertex in the graph. This theorem gives a polynomial-time algorithm for obtaining an optimal edge coloring of a bipartite graph. This algorithm is based on the maximum bipartite matching algorithm implemented in the `GraphTheory` package in the command `BipartiteMatching`. The `ChromaticIndex` command uses this method to obtain optimal edge colorings of bipartite graphs in polynomial time.

These commands along with the `ImportGraph` and `ExportGraph` commands, and definitions of some families of snarks in `SpecialGraphs`, provide a good collection of tools for teaching and research in graph coloring.

5 Networks

There are four commands in the `GraphTheory` package for dealing with networks. They are `IsNetwork`, `RandomNetwork`, `MaxFlow` and `DrawNetwork`.

Here we define a network to be a directed graph with at least one source and one sink. A source is a vertex of the network with no incoming arcs and a sink is a vertex with no outgoing arcs.

The command `IsNetwork` is used to test whether a graph is a network or not. The command can also be used to determine whether a graph with two specified sets of vertices as sources and sinks is a network or not.

```
> N := Digraph({[a,b], [c,d], [b,d], [d,e]});
      N := Graph 1: a directed unweighted graph with 5 vertices and 4 arc(s)
> IsNetwork(N);
      true, {c, a}, {e}
> IsNetwork(N, a, d);
      false
> IsNetwork(N, c, e);
      true
```

5.1 Generating Random Networks

The command `RandomNetwork` can be used to create a random network with specified set or number of vertices. The generated Network has exactly one source and one sink.

We partition the vertices of the network into *levels*. The source and sink are in the first and last levels respectively. If u and v are two vertices in the

network, if there is an arc from u to v and u belongs to the i th level then v must be in the $(i + 1)$ st level.

The user can specify the number of levels in the generated network with an argument p which is a number between 0 and 1. The larger p is, the larger the number of levels will be. For example for $p = 1$ the number of levels would be exactly the number of vertices and the i th level will only has the i th vertex as its member.

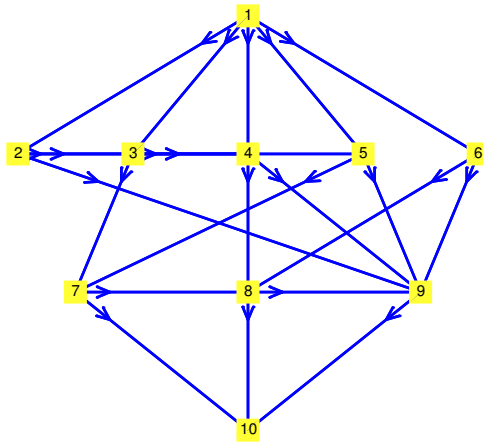
There is also an optional argument q which determines the probability of having an arc between each pair of vertices. The generated network can be forced to be **weighted** or **acyclic** by adding each of these terms in the list of arguments.

```
> N := RandomNetwork(10, 0.3);
> IsNetwork(N);
   true, {1}, {10}
> N := RandomNetwork(10, 0.3, acyclic, weighted);
> IsNetwork(N);
   true, {1}, {10}
> MaxFlow(N, 1, 10);
   1.119382241
```

5.2 Drawing Networks

The **GraphTheory** package includes a command named **DrawNetwork** for displaying networks. The user can specify the set of vertices for sources and sinks of the network, otherwise the sets of **all** possible sources and sinks will be chosen. The user can also choose the network to be displayed either horizontally or vertically.

```
> N := RandomNetwork(10, 0.3, .4, acyclic);
   N := Graph 2 : a directed unweighted graph with 10 vertices and 34 arc(s)!
> DrawNetwork(N);
```



6 The ImportGraph and ExportGraph Commands

The `ImportGraph` and `ExportGraph` commands provide a way to read graphs from files into the `GraphTheory` package and to output graphs to files in some formats. One format is used in the DIMACS graph coloring challenge. For example the command

```
ExportGraph(SpecialGraphs:-CycleGraph(5), "C5.col", dimacs)
```

generates a file `C5.col` in the current working directory with the following content:

```
c Generated by the Maple GraphTheory package
p edge 5 10
e 2 3
e 3 2
e 1 2
e 2 1
e 3 4
e 4 3
e 1 5
e 5 1
e 4 5
e 5 4
```

This file may be read into the `GraphTheory` package using the command
`ImportGraph("C5.col", dimacs).`

Another format supported by the `ImportGraph` and `ExportGraph` commands is the format used by the Mathematica package “Combinatorica”.

`ExportGraph` also supports export to MetaPost. The resulting file may be compiled using the `mpost` command (in Unix systems, for other systems refer to your `LATEX` manual). Figure 1 shows the final result of the following commands.

```
> G := SpecialGraphs:-DoubleStarSnark();
> ExportGraph(G, "dblstar.mp", metapost);
```

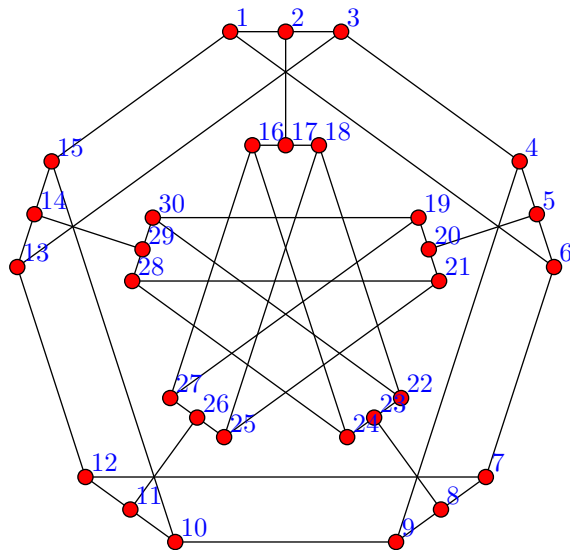


Figure 1: The double-star snark exported to MetaPost

The MetaPost file generated by `ExportGraph` may be edited to customize the result. For example one may remove some vertex labels or change the size of the vertices.

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