# Assignment 2, MACM 204, Fall 2014

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Due Thursday October 2nd at 2:20pm before the beginning of the lab.
Late penalty: -20% for up to 24 hours late. 0 after that.

Please attempt each question in a seperate worksheet.
Print your Maple worksheets and hand in in the MACM 204 drop off box or to me.

There are 10 questions. Attempt all questions.

The purpose of this assignment is to get you to use the 3 - dimensional graphics commands, practice your programming skills, write some recursive programs, compute and visualize partial derivatives, and use Taylor series.

## Question 1

```
> f := x^3-y^3+3*x*y;
```

$$f := x^3 - y^3 + 3\,x\,y$$

The function $f(x, y)$ above has a local minimum ( a valley ) somewhere on the domain $-10 \le x \le 10$ and $-10 \le y \le 10$.
Graph the function using the plot3d command on a suitable domain so that we can see where the local minimum is. To show the local minimum use the patchcontour style. To zoom in on the local minimum, use the view option to restrict the vertical range of the plot. Please also rotate the plot so we can see the local minimum. See the ?plot3d help page.

## Question 2

The roots of the polynomials $x^n - 1$ for $n = 1, 2, 3, \ldots$ are called the roots of unity because they satisfy $x^n = 1$. They have some special properties. One of the properties is that if $n > 1$ then the roots add up to 0. For example, the roots of $x^4 - 1$ are $+1, -1, +i, -i$ which obviously add up to 0. Let's check this in Maple.

```
> L := [solve(x^4-1=0,x)];
```

$$L := [1, -1, I, -I]$$

```
> add( L[i], i=1..4 );
```

$$0$$

Notice that the imaginary number $i$ is I in Maple. Now your first task is to check this property for $2 \le n \le 11$. Do this using a for loop of the form **for n from 2 to 11 do ... od** .
Your second task is to find out what happens when you multiply the roots. For $n = 4$ the product is -1 as shown below. What is the property for the product of the roots for $n \ge 1$?

```
> L[1]*L[2]*L[3]*L[4];
```

$$-1$$

# Question 3

Peter Borwein has a fractal image on his home page [under Vis Number Theory if you care to look.] He created it by computing the complex roots of lots of polynomials then plotting them. The polynomials are special in that the coefficients are restricted to be +1, -1, or 0 only e.g. $x^6 + x^5 - x^3 + x^2 - 1$ . He simply graphed the complex roots in the complex plane and out popped a beautiful fractal image. It was totally unexpected. Try to do this in Maple.

To create a random polynomial of degree *d* with +-1 and 0 coefficients use

```
poly := Randpoly(d,x) mod 2;
```

To compute the complex roots of a polynomial use this command

```
fsolve( poly, x, complex );
```

To plot $n$ complex numbers $z_1, z_2, \ldots, z_n$ in the plane you can use this command

```
plots[complexplot]( [z1,z2,...,zn], style=point );
```

For example

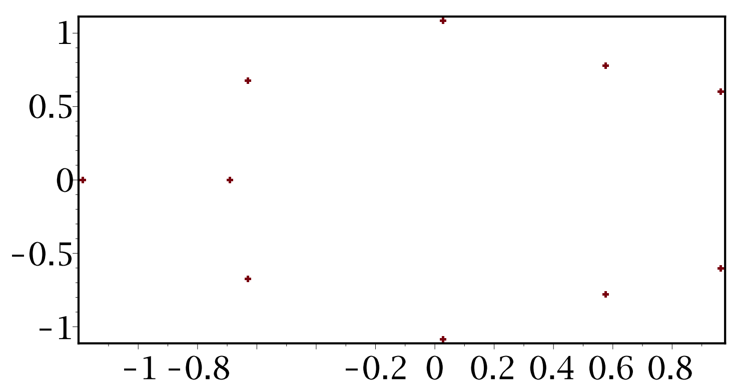
```
> poly := Randpoly(10,x) mod 2;
```

$$poly := x^{10} + x^7 + x^5 + x^4 + x^3 + x + 1$$

```
> R := [fsolve(poly=0,x,complex)];
```

$R := [-1.18689687007298, -0.690662993958197, -0.630124119042763 - 0.675684490700161\,I, -0.630124119042763 + 0.675684490700161\,I, 0.0282550864758195 - 1.08441824789518\,I, 0.0282550864758195 + 1.08441824789518\,I, 0.576347946261116 - 0.779007822911112\,I, 0.576347946261116 + 0.779007822911112\,I, 0.964301018321414 - 0.602836864014598\,I, 0.964301018321414 + 0.602836864014598\,I]$

```
> plots[complexplot]( R, style=point, symbol=cross, axes=box );
```




Now generate at least 1000 polynomials **of degree 18** in a loop, compute their complex roots and graph them on the same plot. You'll need to create a single list of the roots of all 1000 polynomials, so that's 18,000 complex numbers. Note, to get a good image, 10, 000 polynomials is better. But you may have trouble printing the plot for 10,000 polynomials so just print it for 1000 polynomials.

## Question 4

Consider the function $g(a, x) = a^2 \cdot x \cdot (1 - x) \cdot (1 - a \cdot x + a \cdot x^2)$ . We want to study the solutions of $g(a, x) = x$ for $0 \le a \le 4$ and $0 \le x \le 1$.
One way is to create an implicit plot of $g(a, x) = x$ . Do this using the **implicitplot** command in the **plots** package.

Another way is to graph the function $g(a, x) - x$ in 3 dimensions for $0 \le a \le 4$ and $0 \le x \le 1$ and see where the *z* co-ordinate is 0. To do this visually we can graph the 0 function. Do this using the **plot3d** command (graph $g(a, x) - x$ and 0 on the same plot) . Rotate the plot so that it matches the implicit plot.

## Question 5

Consider the function $f(x, y) = 2 \cdot x^2 + 3 \cdot y^2 - x \cdot y - 4$ .
We want to visualize the partial derivatives at the point $x = 1, y = 1$.
First use Maple to compute the partial derivatives $\frac{\partial}{\partial x} f(1, 1)$ and $\frac{\partial}{\partial y} f(1, 1)$ . You should get 3 and 5 respectively so both slopes are positive.
Now generate a 3 dimensional plot of $f(x, y)$ (using the **plot3d** command) and the curves $f(x, 1)$ and $f(1, y)$ (using the **spacecurve** command) and display all three plots on the same graph (using the **display** command).

## Question 6

Consider the function $f(x, y) = 2 \cdot x^4 + 3 \cdot y^4 - x \cdot y - 4$.
We would like to visualize the tangent plane at $x = 1, y = 1$.

Use Maple to construct the linear Taylor polynomial *T* for $f(x, y)$ about $x = 1$ , $y = 1$.
Graph $f(x, y)$ and $T(x, y)$ on the same plot using the plot3d command.

Now use Maple to construct the quadratic Taylor polynomial Q for $f(x, y)$ about $x = 1, y = 1$. This will be T + terms in $(x - 1)^2$, $(y - 1)^2$ and $(x - 1) \cdot (y - 1)$.
Graph $f(x, y)$ and $Q(x, y)$ on the same graph.
You might use the transparency option for *Q(x,y).*

## Question 7

Given a list **L** of values and a value *x*, write a Maple procedure
position := proc(x,L::list) ... end;
so that position(x,L) returns the position (the index) of the first occurence of *x* in **L**. If *x* is not in **L**, return 0. For example for

```
> L := [1,5,4,6,2,4];
```

$$[1, 5, 4, 6, 2, 4]$$

```
> position(4,L);
```

should return 3.

# Question 8

Suppose we want to construct the Taylor polynomial $P(x)$ of degree $n$ for $f(x)$ about the point $x = a$. We can do this using Maple's taylor command which computes a truncated **Taylor series** for $f(x)$ about $x = a$. For example, to get the Taylor polynomial of degree 3 for $e^x$ about $x = 0$ we compute the Taylor series to order $O(x^4)$ as follows.

```
> taylor( exp(x), x=0, 4 );
```

$$1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + O(x^4)$$

So the Taylor polynomial is $1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$

The general formula for the Taylor polynomial $P(x)$ of degree $n$ is given by

$$P(x) = f(a) + f'(a)\cdot(x-a) + \frac{f''(a)\cdot(x-a)^2}{2} + \ldots + \frac{f^{(n)}\cdot(x-a)^n}{n!}$$

Write a Maple procedure TaylorPolynomial($f$, $x$, $a$, $n$) that uses this general formula to compute $P(x)$ . Test your Maple procedure on the following

```
> TaylorPolynomial( exp(x), x, 0, 5 );
> TaylorPolynomial( sin(x), x, 0, 8 );
> TaylorPolynomial( ln(x), x, 1, 5 );
> TaylorPolynomial( sqrt(x), x, 0, 2 );
```

The last one should produce an error for the last example. Explain why.

# Question 9

Given a function $f(x)$ and an interval $a \le x \le b$ and an integer $n > 0$, the Trapezoidal rule $T_n$ on $n$ intervals approximates the definite integral $\int_a^b f(x)\,dx$ using the formula

$T_n = \frac{h}{2}\cdot(f(a) + 2\cdot f(a+h) + 2\cdot f(a+2\cdot h) + \ldots + 2\cdot f(a+(n-1)\cdot h) + f(b))$ . Write a Maple procedure TrapezoidalRule that takes as inputs $f$, $x$, $a$, $b$, $n$ and outputs $T_n$. Do the numerical calculations at 10 digits (the default). Test your procedure on the following

```
> TrapezoidalRule( sin(x), x, 0, 1,  4 );
  TrapezoidalRule( sin(x), x, 0, 1,  8 );
  TrapezoidalRule( sin(x), x, 0, 1, 16 );
  TrapezoidalRule( sin(x), x, 0, 1, 32 );
```

# Question 10

The Taylor series for $e^x$ is about $x = 0$ is given by $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$

For $|x| \le 0.1$ the term in $x^5$ satisfies $\left|\frac{x^5}{120}\right| < 10^{-7}$ . Since the remaining terms are negligible, if we use the Taylor polynomial $1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$ to approximate $e^x$, then for $|x| \leq 0.1$, we will get seven decimal places of accuracy. But what if $|x| > 0.1$?

For $x > 0.1$ we can use the identity $e^{2 \cdot x} = (e^x)^2$ . We first compute $y := e^{\frac{x}{2}}$ then compute $y^2$ to get $e^x$. Now if $\frac{x}{2}$ is still bigger than 0.1 we can divide by 2 again.

For $x < -0.1$ we can use the identity $e^{-x} = \frac{1}{e^x}$ . Putting all of this together we can compute $e^x$ for all x.

Write a Maple procedure `EXP := proc(x::numeric) ... end` that computes $e^x$ accurately to 7 decimal places for $-\infty < x < \infty$. By the way, this is the method your calculator uses to compute $e^x$ using arithmetic only i.e. addition, subtraction, multiplication and divsion.