Assignment 4, MACM 204, Fall 2016

Due Thursday November 10th at 3:00pm.
Late penalty: -20% for up to 24 hours late. 0 after that.

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Please attempt each question in a seperate worksheet (so that you don't destroy your previous work). Print your Maple worksheets (you may print double sided if you wish) and hand them in to me.

Question 1

The Maple command `sum( f(i), i=a..b )` computes a formula for the $\sum_{i=a}^{b} f(i)$.

For example, the sum of the first n integers is $\sum_{i=0}^{n} i$ is given by.

```
> sum( i, i=0..n );
```

$$\frac{1}{2}(n+1)^2-\frac{1}{2}n-\frac{1}{2}$$

The output is not in the usual form. The usual form is

```
> factor( sum(i,i=0..n) );
```

$$\frac{1}{2}n(n+1)$$

Calculate **and simplify** the following 6 sums

(i) $\sum_{i=0}^{n} i^3$ (ii) $\sum_{k=0}^{n} k^2\cdot\binom{n}{k}$ (iii) $\sum_{k=1}^{\infty}\frac{1}{k^2}$ (iv) $\sum_{i=1}^{n} a^i$ (v) $\sum_{i=1}^{n-1}\sum_{j=1}^{i} j\cdot(n-j)$ (vi) $\sum_{i=0}^{\infty}(-1)^i\cdot\frac{x^{2\cdot i+1}}{(2\cdot i+1)!}$

Solution 1

Question 2

Write a Maple procedure GetMaxPrime that takes as input a list of $n>0$ values and returns the largest prime in the list.
If there are no primes then return 0. For example

```
> L := [1,3,5,7,9,5,3,1];
```

$$L := [1, 3, 5, 7, 9, 5, 3, 1]$$

```
> GetMaxPrime(L);
```

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Solution 2

Question 3

This question is related to Newton's law of cooling.
Let $T(t)$ be the temperature of a body of liquid at time t.
Let $Troom$ be the room (ambient) temperature of the surrounding medium (air).
The DE is $T'(t) = k\cdot(Troom - T(t))$ where k is the cooling rate constant.
Solve the differential equation in Maple for Troom = 20 degrees and an initial temperature of 50 degrees.
Given also that $T(20) = 35$, determine k. Now compute T(60). Do all the calculations in Maple.
Finally graph $T(t)$ for $0 \le t \le 100$ together with the room temperature on a suitable domain/range .

Solution 3

Question 4

Carbon 14 decays into Nitrogen 14. Using Google, find the half life *H* of Carbon 14.
The differential equation modeling radioactive decay is

$$y'(t) = -k\cdot y(t)$$

where k is the decay constant and y(0) is the initial concentration of Carbon 14.
Given the half life is *H*, that is, given that $y(H) = \frac{y(0)}{2}$, determine k. You can do this one by hand at first but then do it in Maple.
Solve the DE in Maple and graph the solution for $y(0) = 1$ on a suitable domain.

Solution 4

Question 5

Suppose we have a 400 liter tank. Suppose 8 litres per minute of salt water (brine) flows into the tank at the top and then flows out of the tank at the bottom. Assume for simplicity that the salt water in the tank is stirred so that its concentration is uniform in the tank. Let $S(t)$ be the amount of salt, in grams, in the tank at time t minutes. Suppose the salt water flowing into the tank has concentration 100 grams per liter.

Find the differential equation to model the change in S(t).
Assuming there is no salt in the tank at time t=0 solve the differential equation using Maple.
What is S(∞)? That is, how much salt is in the tank after a long time?
Now graph S(t) for a suitable domain.

Solution 5

Question 6

The logistic growth with harvesting model for a population $y(t)$ at time t is given by

$$y'(t) = \boldsymbol{a} \cdot y(t) \cdot (Y_{max} - y(t)) - H$$

Here $Ymax$ is the maximum sustainable population of the environment, **a** is a constant and *H* is a constant harvesting rate. For Ymax = 8000, a=0.0001, and H = 1000, using the DEplot command, graph y(t) for $0 \le t \le 10$ for the initial values $y(0)$ in 1000, 4000, 8000 and 10000.

Now determine populations *y* for which *y* ' = 0, i.e., find the initial polulations for which there is no growth or decline. You should get two. Graph these on the same graph - you should get two straight lines.

Solution 6

Question 7

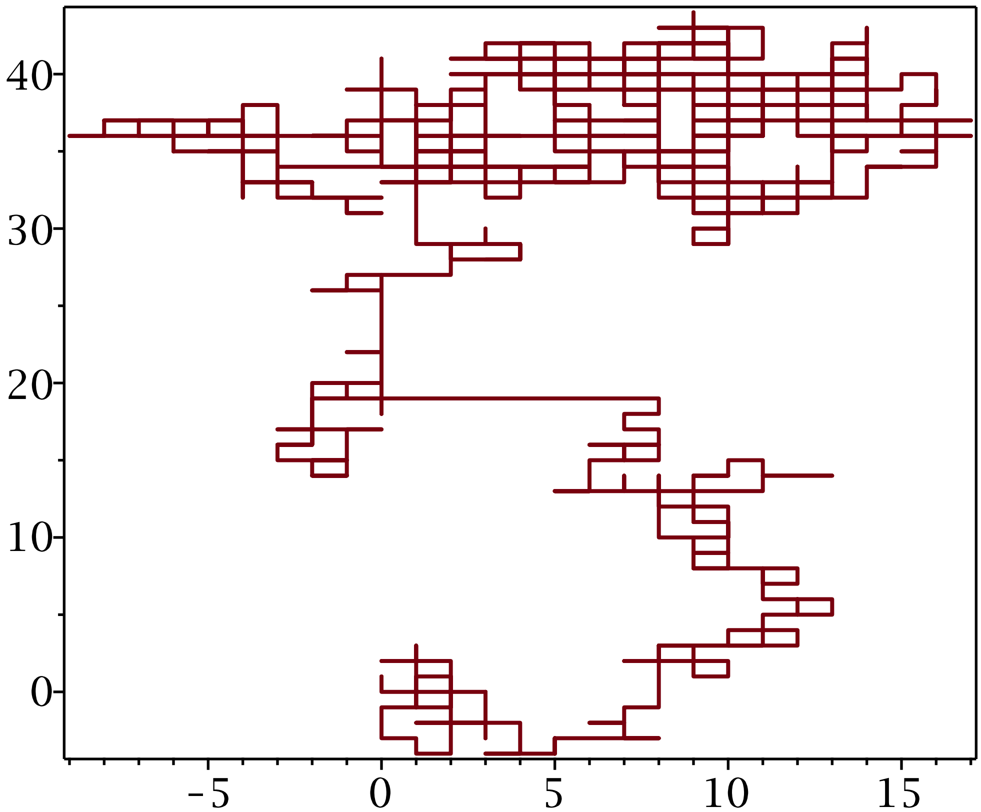
The following is a solution to the random walk exercise in Assignment 3 by a student. It is correct but very slow for large values of n.

```
> R := rand(1..4):
  Walk := proc(n) local x,y,i,P,Q;
    x := 0;
    y := 0;
    Q := [[0,0]];
    for i from 1 to n do
        P := R();
       if P=1 then x := x+1;
       elif P=2 then x := x-1;
       elif P=3 then y := y+1;
       else y := y-1;
        fi;
       Q := [op(Q),[x,y]];
    od;
    Q;
  end:
```

```
> Walk(3);
```

$$[[0, 0], [1, 0], [2, 0], [1, 0]]$$

```
> plot( Walk(1000), style=line, axes=box );
```



To measure the efficiency of the Maple procedure we need to time it for different values of *n*. To time it use the Maple time(); command as follows

```
> start := time(); Walk(n): timetaken := time()-start;
```

Input the above program and time how long this code takes to execute for n=4000, n=8000, n=16000, n=32000.
Interpret the timings you get. Is the running time of the Walk procedure linear in n, quadratic in n, cubic in n, or something else?

The problem is the code `Q := [op(Q),[a,b]]` which appends the next point to the list . Doing this in a loop means that we are creating a list of length 2 then 3 then 4 then 5 and so on. Creating lists of length 1000 takes time. This makes the procedure slow. One solution is to store the points in an Array instead of a list.
To create an Array of n values indexed from 1 to n use

```
> A := Array(1..n);
```

To insert a value into the Array use

```
> A[1] := [0,0];
```

Inserting a value into an array is fast even if the array is very large. Since Maple's plot(A,style=line) command accepts for A either a list of points or an Array of points, we don't need to change anything else

Now reprogram the Walk procedure to use an Array and retime the improved code for n=4000, n=8000, n=16000, n=32000, n=64000. It should make a huge difference.

Solution 7

Question 8

The square of the error of the line $y = m \cdot x + b$ fitting n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ is given by

$$E(m, b) = \sum_{i=1}^{n} (y_i - (m \cdot x_i + b))^2 ..$$

Input this sum into Maple as follows

```
> restart;
  E := sum( (y[i]-(m*x[i]+b))^2, i=1..n );
```

Calculate the partial derivativives
$E_m(m, b), E_b(m, b), E_{mm}(m, b), E_{bb}(m, b), E_{mb}(m, b)$ using Maple .
The least-squares solution (M, B) is a solution of $\{E_m(m, b) = 0, E_b(m, b) = 0\}$, i.e., it is a critical point.
To show that it is a local minimum we need to show that $E_{mm}(M, B) > 0$ and $E_{bb}(M, B) > 0$. Do that.

We also need to show that the critical point is is not a saddle.
To do that we show that $\mathrm{D}(M, B) > 0$ where $\mathrm{D}(m, b) = E_{mm} \cdot E_{bb} - E_{mb}^2$.
This is outside of the scope of the course but you can at least come up with the inequality that needs to be proven.

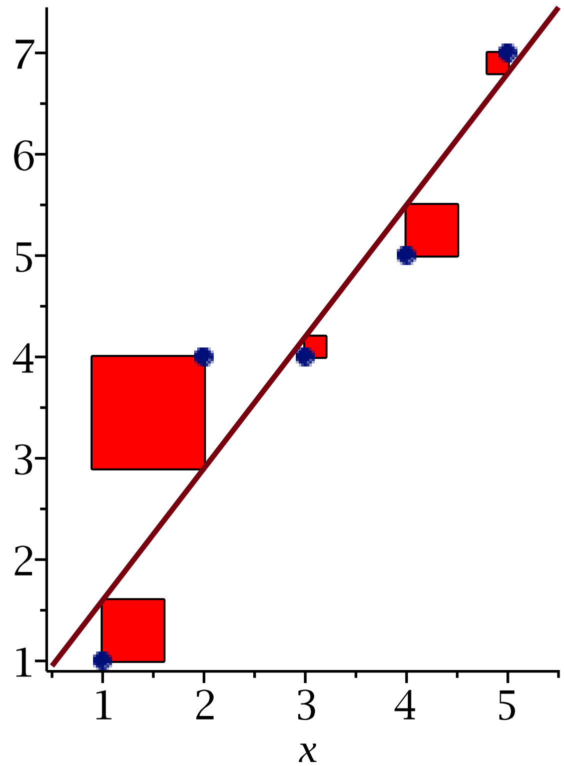
Solution 8

Question 9

Consider the 5 data points (1,1), (2,4), (3,4), (4,5), (5,7).
Find the line $y = m \cdot x + b$ that fits the data in the least squares sense.
You may use the LeastSquares command from the CurveFitting package.

Calculate the least squares error $\sum_{i=1}^{n} (y_i - (m \cdot x_i + b))^2$.

Now generate a plot of the data points, the line and squares showing the least square error as shown in the following figure.



▶ **Solution 9**