Assignment 5, MACM 204, Fall 2016

Michael Monagan.

Due Tuesday November 29th at 3:00pm.
Late penalty: -20% for up to 48 hours late. 0 after that.

There are 8 questions. Questions 5 and 8 are review questions to prepare you for the final. Attempt all questions.

Question 1

Consider a chemical reaction involving chemicals A, B, and C in which A is converted to B at a rate k_1 and chemical B is converted to C at a rate of k_2 as illustrated in the compartment model below. I created the figure in Maple using the "Canvas" option under the insert menu, which I didn't know about but is very useful. Letting $A(t), B(t), C(t)$ be the amount of chemical A, B, C at time t we can model the chemical reactions with the differential equations

$A'(t) = -k_1 \cdot A(t)$,
$B'(t) = k_1 \cdot A(t) - k_2 \cdot B(t)$,
$C'(t) = k_2 \cdot B(t)$.

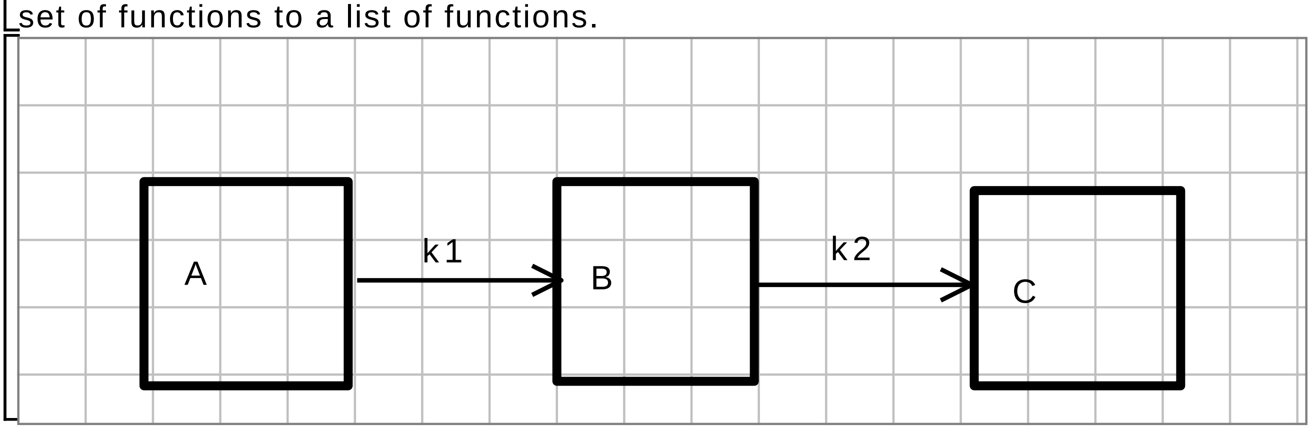
Part (a): Solve the differential equations using `dsolve` for the initial value $A(0) = N, B(0) = 0, C(0) = 0$. Here $N > 0$ is the initial amount of chemical A. Try to simplify the solutions as best you can.

You will see that there is a factor of $k_1 - k_2$ in the denominators which means the solutions are not valid for the case $k_1 = k_2$. So solve the differential equations for this case too. Again, try to simplify the solutions.

Part (b): Now, looking at the model, can you predict what $A(\infty), B(\infty), C(\infty)$ are? You don't need to do any math to figure this out. For the case $k_1 = k_2$ try also taking the limit of $A(t), B(t), C(t)$ as $t \to \infty$. You will need to tell Maple that $k_2 > 0$. You can do this using the assuming statement like this

```
> limit( A(t), t = infinity ) assuming k2>0;
```

Part(c): For $N = 5, k_1 = 0.2, k_2 = 0.1$, solve the differential equations and graph $A(t), B(t), C(t)$ verses t on the same graph for a suitable domain using the plot command. Attach a legend = ["A","B","C"]. To do this you need to convert from a set of functions to a list of functions.



Solution 1

Question 2

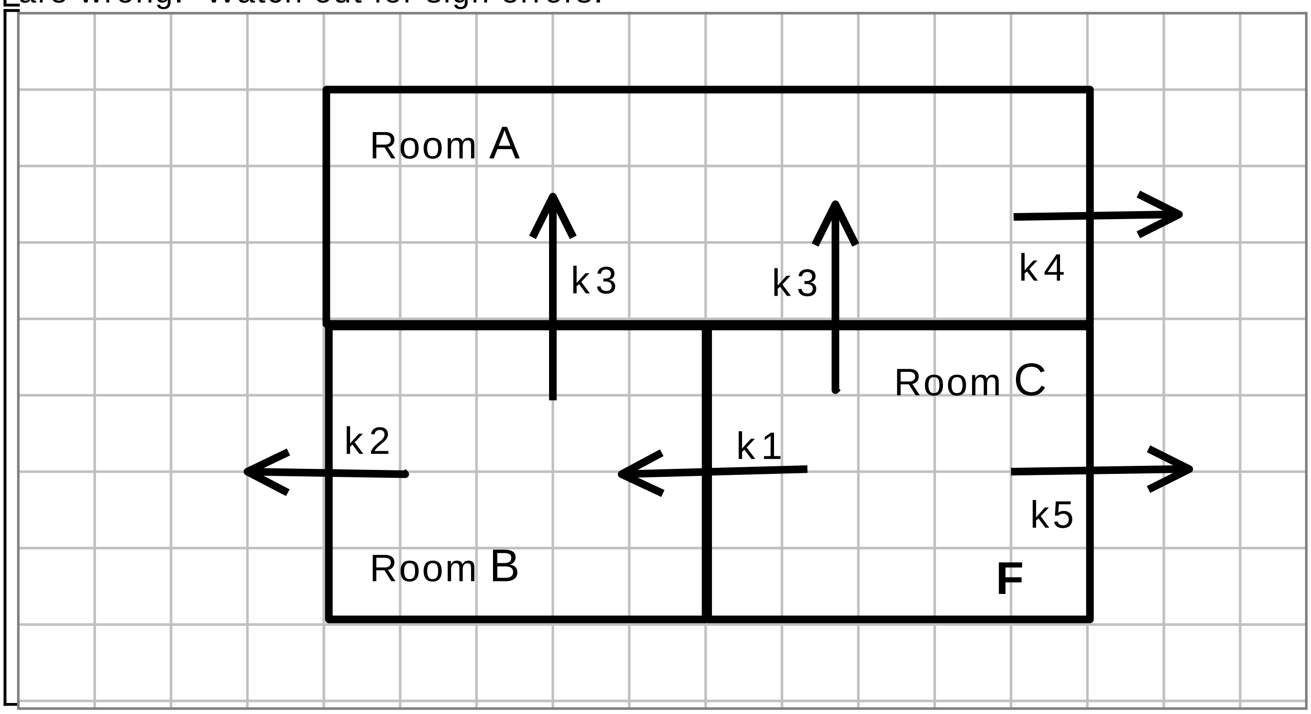
Shown in the figure below is a house with three rooms, A, B and C. Rooms B and C are the same size and shape. There is a furnace **F** in room C which heats room C. Let $A(t)$, $B(t)$, $C(t)$ be the temperature at time t in rooms A, B, C respectively and let Am be the outside temperature. Shown in the figure are the cooling rate constants k_1, k_2, k_3, , k_4, k_5 for how heat moves through the walls of the house.

Part (a) Write down a system of 3 differential equations $A'(t) = ...$, $B'(t) = ...$, $C'(t) = ...$ for the house.

Part (b) Determine the temperature equilibrium point as a function of F, Am, $k1$, $k2$, $k3$, $k4$, $k5$. Try to simplify the formulas by writing them in the form $Am + f(k1, k2, k3, k4, k5) \cdot F$.

Part (c) For $F = 5$, $Am = 0$, $k1 = 0.3$, $k2 = 0.1$, $k3 = 0.3$, $k4 = 0.2$, $k5 = 0.1$ determine the temperature equilibrium, solve the differential equations using dsolve, and compute the limit of $A(t)$, $B(t)$, $C(t)$ as $t \to \infty$.

Part (d) Finally, graph $A(t)$, $B(t)$, $C(t)$ on the same graph on a suitable domain. Note, if the graph looks wrong this could be because your differential equations are wrong. Watch out for sign errors.



Solution 2

Question 3

Shown in the figure below is lake Erie and lake Ontario and the main rivers flowing through them (the arrows). Google says that the volume of lake Erie is about $500 \cdot km^3$ and lake Ontario is about $1500 \cdot km^3$ and the amount of water flowing through the lakes is about $60 \cdot km^3$ per year. Yes, that's kilometers cubed. The goal is to model the amount of pollution in the two lakes at time *t* (years). We will assume that initially, there is no pollution in either lake and that the river flowing into lake Erie is polluted and is bringing in 30 tons of pollutant per year.

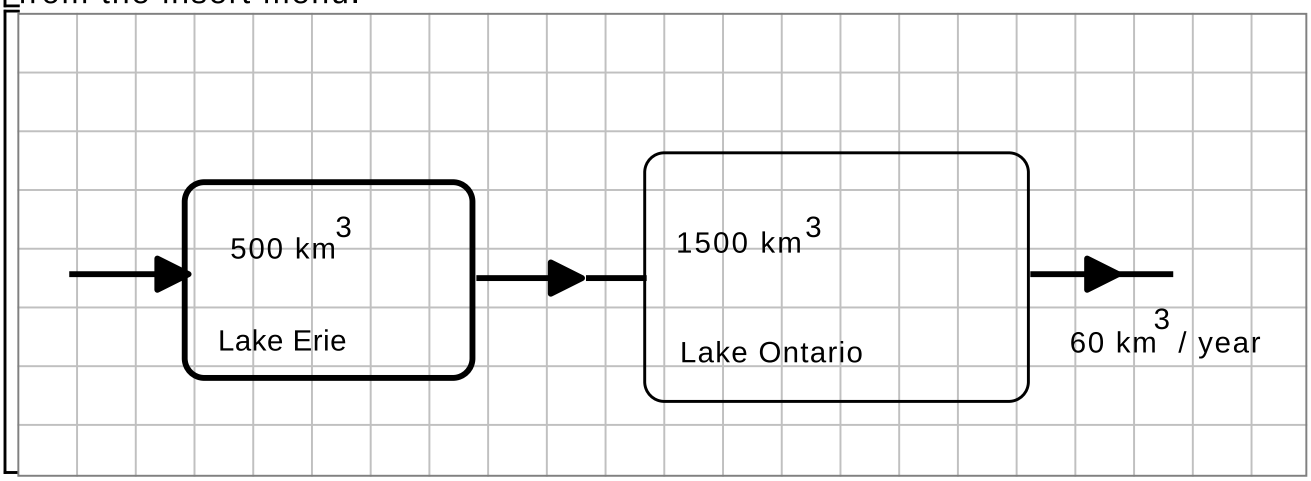
Let $Er(t)$ be the amount of pollutant (in tons) in lake Erie at time *t* and let $On(t)$ be the amount of pollutant (in tons) in lake Ontario at time *t* (years).

Part (a) Set up two differential equations, one for the amount of pollution in lake Erie at time *t* and the other for the amount of pollution in lake Ontario at time *t*. This problem is very much like the tank problem in the last assignment, except that here we have two tanks (two lakes).

Parb (b) Solve the differential equations and plot the solutions for a suitable time domain. You should see that the amount of pollutant in each lake increases from 0 to a maximum. What are the maximums?

Part (c) Using the DEplot command in the DEtools package, generate a field plot with solution curves for initial values $Er(0) = 0$, $On(0) = 0$ and $Er(0) = 500$, $On(0) = 0$ and $Er(0) = 500$, $On(0) = 1500$ on the same plot.

Part (d) Reproduce the figure below in Maple. It's a "Canvas" which you can create from the insert menu.



▶ Solution 3

▼ Question 4

The Kermack-McKendrick virus spread model (where we partition the individuals in a population into those which are susceptible, infected, and recovered), is given by

$S'(t) = -\beta \cdot S(t) \cdot I(t),\ I'(t) = \beta \cdot S(t) \cdot I(t) - \alpha \cdot I(t),\ R'(t) = \alpha \cdot I(t)$

where $S(t), I(t), R(t)$ is the proportion of the population which is susceptible, infected and recovered at time t. We cannot solve the differential equations for nice formulas but we can solve then numerically for a given set of initial values.

Part (a) For $\beta = 0.3$, $\alpha = 0.1$, $S(0) = 0.99$, $I(0) = 0.01$, $R(0) = 0.00$ solve the differential equations using dsolve(..., numeric) and plot the solutions for $S(t), I(t), R(t)$ on the same plot using the odeplot command in the plots package. You should see an epidemic, that is, $I(t)$ increases to a maximum and then drops down to 0. Please include a decent legend on the plot. Looking at the plot, what proportion of the population never becomes infected, i.e., what is $S(\infty)$? [Rough estimate from the plot is fine.]

Five years ago the H1N1 flu virus hit British Columbia. Perhaps you got vaccinated? I did. Dr. Perry Kendall was the provincial health officer then. He was advising us that we should all get a vaccination, not just to protect us individually, but to prevent H1N1 from becoming an epidemic. How can that work? The idea is that if a high enough proportion of the population gets the vaccination then they cannot contract the virus, so the virus cannot spread as easily through the population, and this can prevent an epidemic (outbreak). We can test this hypothesis in the SIR virus model. Suppose 50% of the population gets the H1N1 vaccine and 1% has the H1N1 virus (so 49% is susceptible). We can simulate this in the model using $S(0) = 0.49, I(0) = 0.01, R(0) = 0.50.$

Parb (b) Solve the differential equations using this set of initial values to determine if this makes the virus endemic (die out) or not. If not, find, roughly, the %age of the population that must be vaccinated (this is for $\alpha=0.1$, $\beta=0.3$) to make the virus endemic.

Solution 4

Question 5

Let $M(t)$ be the amount owed on a 30 year mortgage of \$200,000 at time *t* years.
Suppose the annual interest rate on the mortgage is *r*=4%.
Suppose the term of the mortgage is 30 years, i.e., M(30) should be 0.
Suppose we pay \$*P* per year.

We can model the change in what we owe the bank with the differential equation
$M'(t) = r \cdot M(t) - P$ dollars **per year**
and initial values $M(0) = \$200000.$
This model assumes the interest is charged continuously (banks usually charge interest daily which is approximately continuous) and if we assume we make the payments continuously (banks usually require us to pay monthly or weekly which is approximately continuous over 30 years). So the values we obtain with it will be approximations.

```
> restart;
  de := diff( M(t),t) = r*M(t)-P;
  r := 0.04;
```

$$de := \frac{d}{dt} M(t) = r\, M(t) - P$$

$$r := 0.04$$

```
> sol := dsolve( {de, M(0)=200000}, M(t) );
```

$$sol := M(t) = 25\, P + e^{\frac{1}{25}\, t} (200000 - 25\, P)$$

To determine the anual payment P we solve $M(30) = 0$ for P.

```
> eq := eval( rhs(sol), t=30.0 ) = 0;
```

$$eq := -58.00292308\,P + 6.640233846\,10^{5} = 0$$

```
> annualpayment := solve( eq, P );
```

$$annualpayment := 11448.10208$$

So we pay $11,448.10 dollars per year. If we are making monthly payments we would pay

```
> monthlypayment := annualpayment/12;
```

$$monthlypayment := 954.0085067$$

Also, the total interest we pay is quite a bit

```
> interestpaid := 30*annualpayment - 200000;
```

$$\$143{,}443.06$$

Suppose you want to pay off the mortgage faster to avoid paying all that interest!! Suppose you decide to pay P = $15,448 pear year instead (so $4000 more per year).

Part (a) Solve the differential equation with r = 4% per year and P = 15,448 per year and determine (i) how long it will take you to pay off the mortgage and (ii) how much interest you will pay.

Part (b) Now graph (**on the same plot**) *M(t)* for *P* = $11,448.10 **and** *M(t)* with *P* = $15,448 on a suitable domain. Include a legend and title.

Solution 5

Question 6

In Question 5 the 30 year mortgage was $200,000 and the interest rate *r* was 4% per year compounded daily.
We modelled the change in what we owe the bank with the differential equation

$M'(t) = r \cdot M(t) - P$ dollars per year

We solved the differential equation with initial value $M(0) = 200000$.
We worked out if we set $P = \$11448.10$ per year then $M(30) = 0$.

But the differential equation assumes a continuous process in which we are continually being charged interest and we are continually paying down the mortgage. In reality the bank charges us interest once a day and suppose we make payments once a month at the end of the month. Assuming no leap years for simplicity, over a 30 year period there will be $30 \cdot 365 = 10{,}950$ interest charges which is almost continuous. But there are only $30 \cdot 12 = 360$ payments, less continous.
So the actual process is a discrete process. The goal of this question is to simulate these interest charges and payments. We will compute the actual ammount owed after 30 years assuming we get charged 0.04/365 % interest every day and we pay $11448.10/12 per month on the last day of the month. So starting with

```
> M := 200000; # mortgage ammount -- what we currently owe
```

```
r := 0.04/365; # interest per day
P := 11448.10/12; # payment per month
```

$$M := 200000$$

$$r := 0.0001095890411$$

$$P := 954.0083333$$

write Maple code that simulates, for 30 years, the daily interest charges and monthly payments. You must determine how many days are in each month (assume no leap years) so that the payments are made on the last day of each month.

If you do this correctly, you should find that after 30 years we still owe about $1000 which is about one months payment. The discrepency is partly because we are paying at the end of the month. Repeat the simulation with payments made on the 15th day of each month.

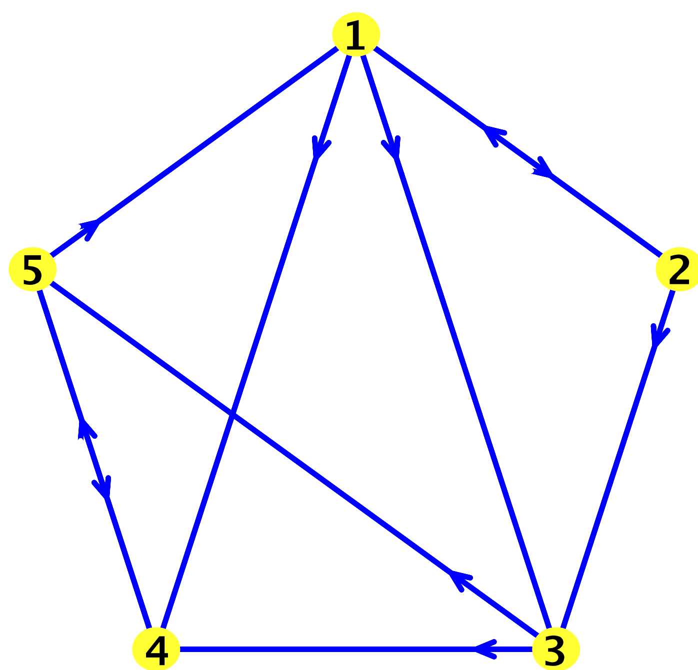
Solution 6

Question 7

```
> with(GraphTheory):
> G := Graph( {[1,2],[1,3],[1,4],[2,1],[2,3],[3,4],[3,5],[4,5],
  [5,1],[5,4]}, directed );
```

$G :=$ *Graph 1: a directed unweighted graph with 5 vertices and 10 arc(s)*

```
> DrawGraph(G)
```



Shown above is a directed graph with 5 vertices. The graph represents 5 web pages and the directed edges represent the hyperlinks between the web pages. So we can get from page 1 to pages 2, 3 and 4, from page 2 to pages 1 and 3, from page 3 to pages 4 and 5, from page 4 to page 5 and from page 5 to pages 1 and 4. We can encode this information using

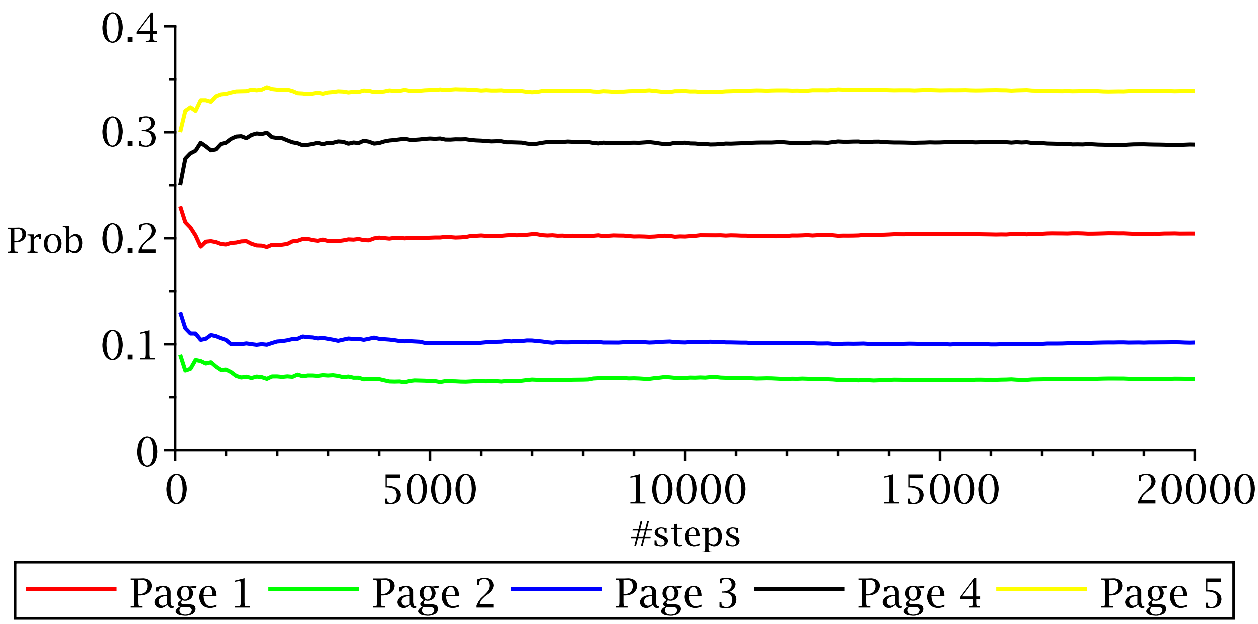
```
> P := Array(1..5):
  P[1] := [2,3,4];
  P[2] := [1,3];
  P[3] := [4,5];
  P[4] := [5];
  P[5] := [1,4];
```

Part (a)

Now, starting at vertex 1, simulate a random walk of length $N = 20,000$ steps through the graph. Store the walk in an array W. So since vertex 1 is connected to vertex 2 and 4, the random walk will go to vertex 2 or 4 with probability 0.5. So if $W_1 = 4$ this means after the first random step you are on page 4. At the end of the random walk, for each vertex, calculate the probability the walker visits that vertex..

Part (b)

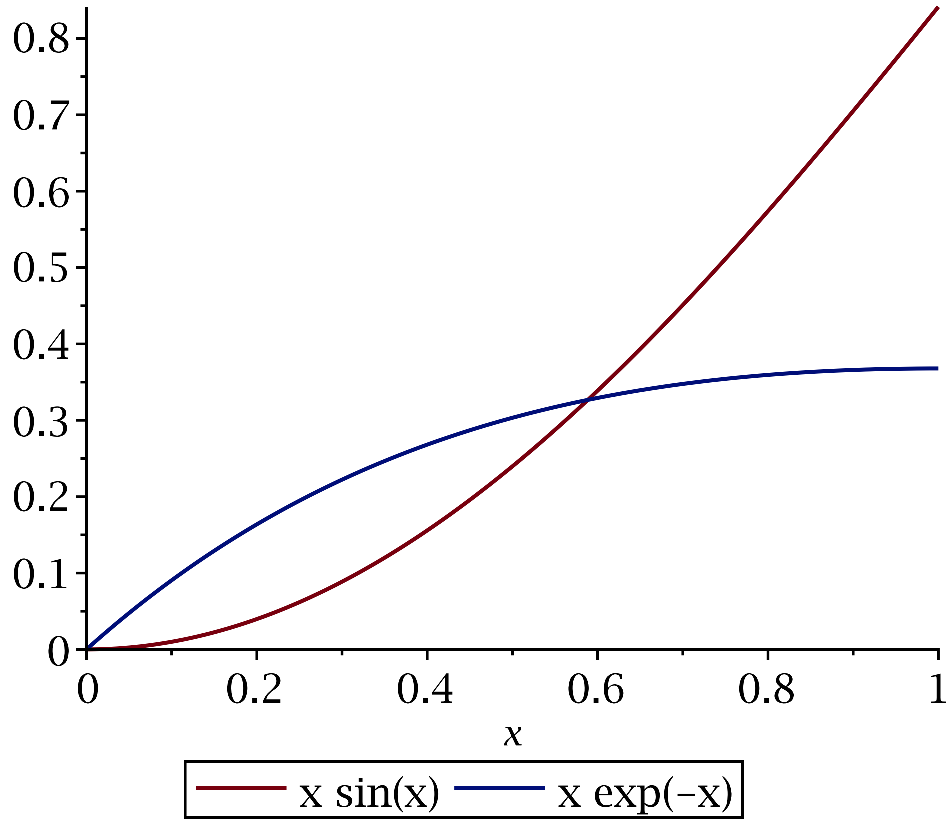
Now modify your code to do the following. After each 100 steps, for each vertex, calculate the probability that the walker was on that vertex. For each vertex, this generates a sequence of 200 probabilities that should converge.
Graph these 5 sequences on a single graph. Include a legend. You should get a picture like the following.



▶ Solution 7

▼ Question 8

Generate the plot below of the two functions $f(x) = x \cdot \sin(x)$ and $g(x) = x \cdot e^{-x}$ on the domain $0 \le x \le 1$. Calculate the area *A* enclosed between the two curves as a definite integral. You will run into a difficulty when you try to solve $f(x) = g(x)$ using the solve command or if you try to do it by hand. I leave it up to you to decide what to do.



Solution 8