Assignment 6, MACM 204, Fall 2026

Michael Monagan.

Due 11pm Wednesday December 2nd. **Late penalty:** -10% per hour late.

Please answer each question in a separate Maple worksheet so you can upload them to Crowdmark easily.

Question 1

Consider the three limits

```
> limit(sin(x)/x, x=0);
```

```
> limit(x*ln(x), x=0, right);
```

```
> limit(x^2/exp(x), x=infinity);
```

All three limits can be calculated using l'Hoptital's rule. I need to use l'Hopital's rule each time I teach MATH 152, the integral calculus course. However, the easiest way to determine a limit is to graph the function near the limit point. Do this. Now compute the limits using Maple. Do this either using the limit command (see ?limit) or using the 2D input and the limit command under the Calculus pallete.

Solution 1

Question 2

Consider a chemical reaction involving chemicals A, B, and C in which A is converted to B at a rate k_1 and chemical B is converted to C at a rate of k_2 as illustrated in the compartment model below. I created the figure in Maple using the "Drawing" option under the Insert menu. Letting $A(t), B(t), C(t)$ be the amount of chemical A, B, C at time t we can model the chemical reactions with the differential equations

$$A'(t) = -k_1 \cdot A(t),$$
$$B'(t) = k_1 \cdot A(t) - k_2 \cdot B(t),$$
$$C'(t) = k_2 \cdot B(t).$$

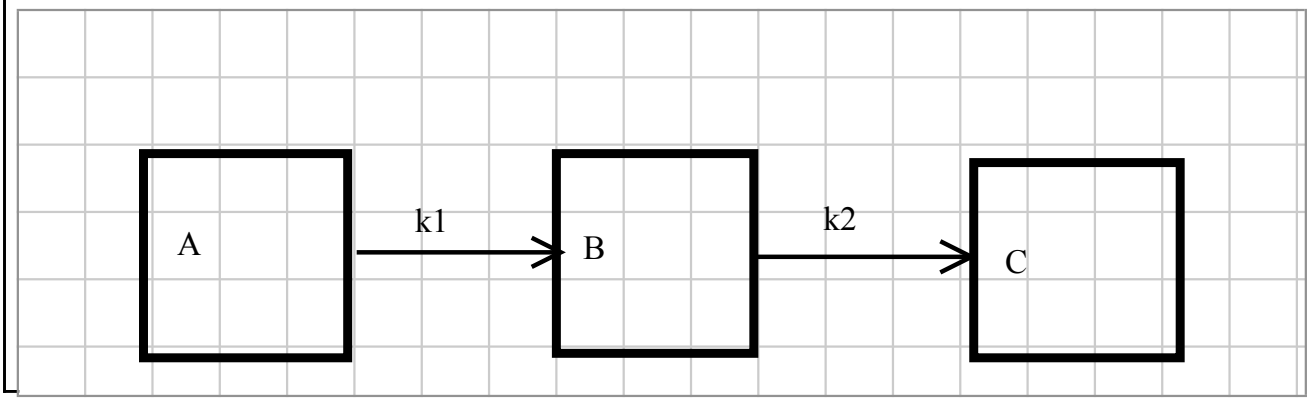
Part (a): Solve the differential equations using dsolve for the initial value $A(0) = N, B(0) = 0, C(0) = 0$. Here $N > 0$ is the initial amount of chemical A. Try to simplify the formulas for the solutions as best you can.

You will see that there is a factor of $k_1 - k_2$ in the denominators which means the solutions are not valid for the case $k_1 = k_2$. So solve the differential equations for this case too. Again, try to simplify the solutions.

Part (b): Now, looking at the model, can you predict what $A(\infty), B(\infty), C(\infty)$ are? You don't need to do any math to figure this out. For the case $k_1 = k_2$ try also taking the limit of $A(t), B(t), C(t)$ as $t \to \infty$. You will need to tell Maple that $k_2 > 0$. You can do this using the assuming statement like this

```
> limit( A(t), t = infinity ) assuming k2 > 0;
```

Part(c): For $N = 5$, $k_1 = 0.3$, $k_2 = 0.1$, solve the differential equations and graph $A(t), B(t), C(t)$ verses t on the same graph for a suitable domain using the plot command. Attach a legend = ["A", "B","C"]. To do this you need to convert from a set of functions to a list of functions.



Solution 2

Question 3

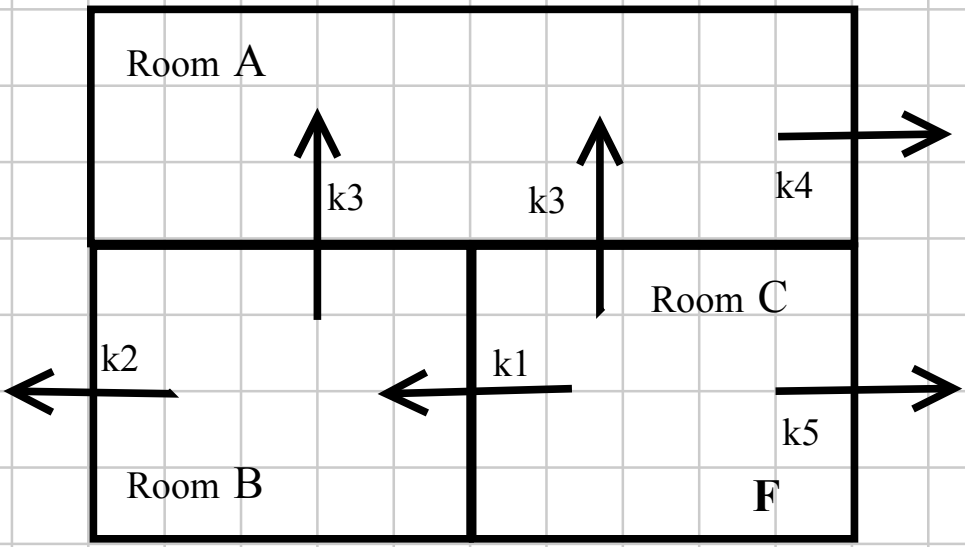
Shown in the figure below is a house with three rooms, A, B and C. Rooms B and C are the same size and shape. There is a furnace **F** in room C which heats room C. Let $A(t), B(t), C(t)$ be the temperature at time t in rooms A, B, C respectively and let Am be the outside temperature. Shown in the figure are the cooling rate constants $k_1, k_2, k_3, , k_4, k_5$ for how heat moves through the walls of the house.

Part (a) Write down a system of 3 differential equations $A'(t) = ...,\ B'(t) = ...,\ C'(t) = ...$ for the house.

Part (b) Determine the temperature equilibrium point as a function of $F, Am, k1, k2, k3, k4, k5$. Try to simplify the formulas by writing them in the form $Am + f(k1, k2, k3, k4, k5)\cdot F$. Use the collect command to do this.

Part (c) For $F = 5,\ Am = 0, k1 = 0.3, k2 = 0.1, k3 = 0.3, k4 = 0.2, k5 = 0.1$ determine the temperature equilibrium using the solution to part (b) and also solve the differential equations using dsolve, and compute the limit of $A(t), B(t), C(t)$ as $t \to \infty$.

Part (d) Finally, graph $A(t), B(t), C(t)$ on the same graph on a suitable domain. Note, if the graph looks wrong this could be because your differential equations are wrong. Watch out for sign errors.



Solution 3

Question 4

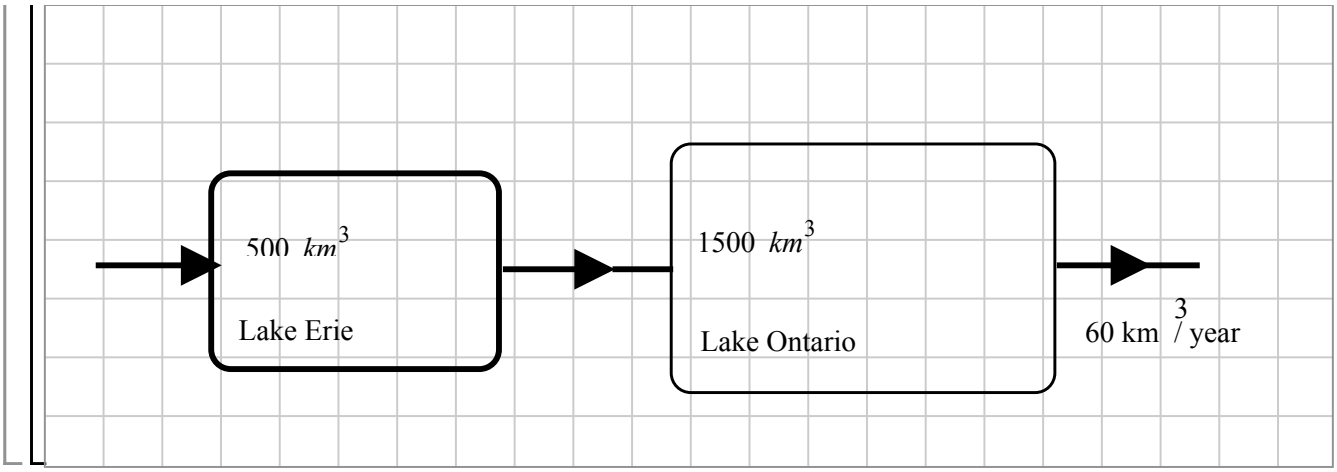
Shown in the figure below is lake Erie and lake Ontario and the main rivers flowing through them (the arrows). Google says that the volume of lake Erie is about $500 \cdot km^3$ and lake Ontario is about $1500 \cdot km^3$ and the amount of water flowing through the lakes is about $60 \cdot km^3$ per year. Yes, that's kilometers cubed. The goal is to model the amount of pollution in the two lakes at time t (years). We will assume that initially, there is no pollution in either lake and that the river flowing into lake Erie is polluted and is bringing in 30 tons of pollutant per year.

Let $Er(t)$ be the amount of pollutant (in tons) in lake Erie at time t and let $On(t)$ be the amount of pollutant (in tons) in lake Ontario at time t (years).

Part (a) Set up two differential equations, one for the amount of pollution in lake Erie at time t and the other for the amount of pollution in lake Ontario at time t. This problem is very much like the tank problem in the last assignment, except that here we have two tanks (two lakes).

Parb (b) Solve the differential equations and plot the solutions for a suitable time domain. You should see that the amount of pollutant in each lake increases from 0 to a maximum. What are the maximums?

Part (c) Using the DEplot command in the DEtools package, generate a field plot with solution curves for initial values $Er(0) = 0$, $On(0) = 0$ and $Er(0) = 500$, $On(0) = 0$ and $Er(0) = 500$, $On(0) = 1500$ on the same plot.



▶ Solution 4

▼ Question 5

The Kermack-McKendrick virus spread model (where we partition the individuals in a population into those which are susceptible, infected, and recovered), is given by

$S'(t) = -\beta \cdot S(t) \cdot I(t),\ I'(t) = \beta \cdot S(t) \cdot I(t) - \alpha \cdot I(t),\ R'(t) = \alpha \cdot I(t)$

where $S(t), I(t), R(t)$ is the proportion of the population which is susceptible, infected and recovered at time t. We cannot solve the differential equations for nice formulas but we can solve then numerically for a given set of initial values.

Part (a) For $\beta = 0.2$, $\alpha = 0.1$, $S(0) = 0.99$, $I(0) = 0.01$, $R(0) = 0.00$ solve the differential equations using dsolve(..., numeric) and plot the solutions for $S(t), I(t), R(t)$ on the same plot using the odeplot command in the plots package. You should see an epidemic, that is, $I(t)$ increases to a maximum and then drops down to 0. Please include a decent legend on the plot. Looking at the plot, what proportion of the population never becomes infected, i.e., what is $S(\infty)$? [Rough estimate from the plot is fine.]

Six years ago the H1N1 flu virus hit British Columbia. Perhaps you got vaccinated? I did. Dr. Perry Kendall was the provincial health officer then. He was advising us that we should all get a vaccination, not just to protect us individually, but to prevent H1N1 from becoming an epidemic. How can that work? The idea is that if a high enough proportion of the population gets the vaccination then they cannot contract the virus, so the virus cannot spread as easily through the population, and this can prevent an epidemic (outbreak). We can test this hypothesis in the SIR virus model. Suppose 25% of the population gets the H1N1 vaccine and 1% has the H1N1 virus (so 74% is susceptible). We can simulate this in the model using

$S(0) = 0.74, I(0) = 0.01, R(0) = 0.25.$

Parb (b) Solve the differential equations using this set of initial values to determine if this makes the virus endemic (die out) or not. If not, find the %age of the population that must be vaccinated (this is for α=0.1, β=0.2) to make the virus endemic.

▶ Solution 5

Copyright, Michael Monagan, November 2024

▼ Question 6

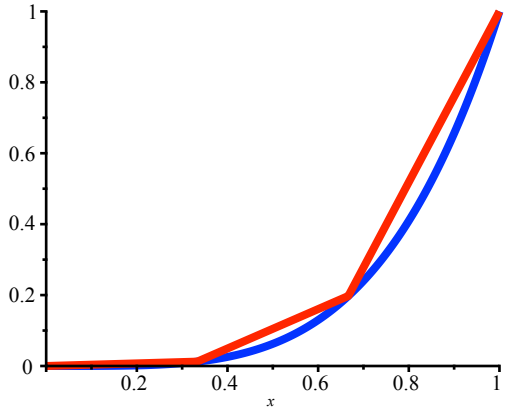
Given a function $y = f(x)$, the formula for the arc length L of the $f(x)$ from $x = a$ to $x = b$ is given by

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Part (a) For $y = x^n$ calculate the length L for $n = 2, 3, 4, 5, 6, 7, 8$ exactly using Maple's integration command. Use the evalf command to also get decimal approximations for the integrals. For $n = 4$ I get 1.600229428.

Part (b) You should get an answer involving a hypergeometric function for $n = 4, 5, 6, 7, 8$. Guess a formula for L as a function of n and verify that your formula is correct for $n = 2$ **and** $n = 3$.

Part (c) I'd like you to code the obvious numerical method to approximate the length L.
Shown below is a plot of $y = x^4$ and three line segments. The length of the three line segments is an approximation of $L = 1.600229428$. The more line segments, the better the approximation. Write a Maple procedure ARCLEN(f, x, a, b, n) that calculates the length of n line segments. The x coordinates should be equally spaced on $[a, b]$.



Test your procedure on this data.

```
> ARCLEN(x^4,x,0,1,3);
```

1.583827901 **(11.1)**

```
> ARCLEN(x^4,x,0,1,1000);
```

1.600229278 **(11.2)**

Part (d)
Write a Maple procedure PLOTARCLEN(f, x, a, b, n) that creates the plot. Test it on

```
> PLOTARCLEN(x^4,x,0,1,3);
> PLOTARCLEN(x^4,x,0,1,4);
```

▶ Solution 6