Theorem 12.1 (the Risch Structure Theorem)

Let K be a field of constants, $F_0 = K(x)$, and $F_n = K(x)(\theta_1, \ldots, \theta_n)$ where θ_j is either

(i) algebraic over $F_{j-1} = F_0(\theta_1, \ldots, \theta_{j-1})$ or
(ii) $\theta_j = \log u_j$ for some $u_j \in F_{j-1}$ or
(iii) $\theta_j = e^{w_j}$ for some $w_j \in F_{j-1}$.

Then (i) $h = e^g$ where $g \notin K$ is algebraic over F_n iff $\exists\, c_i \in \mathbb{Q}$ s.t.

$$g + \sum_i c_i w_i \in K$$

and (ii) $h = \log f$ where $f \notin K$ is algebraic over F_n iff $\exists\, k_j \in \mathbb{Z}$, $k_0 \neq 0$ s.t.

$$f^{k_0} \cdot \prod_{j \neq 0} u_j^{k_j} \in K$$

Given $\int f(x)\,dx$, apply the theorem to construct F_n s.t. $f(x) \in F_n$, $\theta_i \notin F_{i-1}$ and transcendental (θ_i not also algebraic).

E.g. $e^{\frac{1}{2}x}$ and e^{2x} are algebraic over $\mathbb{Q}(x)(\theta_1 = e^x)$
$\log x^{-1}$ and $\log x^2$ are algebraic over $\mathbb{Q}(x)(\theta_1 = \log x)$