Theorem 12.4 (Liouville's Principle)

Let F be a differential field with an algebraically closed constant field K e.g. $K = \mathbb{C}$. Let $f \in F$.
$\int f(x) \, dx \in G$ where G is an elementary extension of F i.e. $G = F(\Theta_1, \ldots, \Theta_n)$ where Θ_i is exp, log, algebraic over $F(\Theta_1, \ldots, \Theta_{i-1})$. Then $\exists \, V_0, V_1, \ldots, V_m \in F$ and constants $c_1, c_2, \ldots, c_m \in K$ such that

$$\int f \, dx = V_0 + c_1 \log V_1 + \cdots + c_m \log V_m$$

i.e. if $\Theta_i \notin F$, then Θ_i is a logarithm.

E.g. $\int x e^x + \dfrac{2}{1+x} \, dx = \underbrace{(x-1)e^x}_{V_0 \in F} + \underset{c_1 \in \mathbb{C}}{2} \log \underset{V_1 \in F}{(1+x)}$

$F = \mathbb{C}(x)(e^x)$ $\qquad\qquad G = F(\Theta_x = \log(1+x))$

E.g. Suppose $\int f \, dx = x e^{x^2}$ and $f \in F$.
L.P. $\Rightarrow e^{x^2} \in F$ and e^{x^2} appears in f.

Differentiating both sides
$$f = [x e^{x^2}]' = 1 \cdot e^{x^2} + 2x^2 e^{x^2} = (1 + 2x^2) e^{x^2}$$
Theorem 12.3 says $\deg(f', e^{x^2}) = \deg(f', e^{x^2})$