Lec24A Logarithmic Case: Rational Part Michael Monagan

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12.5 The Risch Integration Algorithm (Bob Risch, IBM, 1968)

Let $F_n = K(x)(\theta_1, \ldots, \theta_n)$ where K is an algebraically closed constant field, θ_i is an elem. ext. of F_{i-1} (exp, log, alg.) and $\theta_i' \neq 0$. To integrate $f(x) \in F_n$ the Risch algorithm integrates functions in F_{n-1} recursively. So the recursion base is $F_0 = K(x)$ i.e. Ch 11.

The transcendental case: $\theta_i = \log u_i$ or $\theta_i = e^{w_i}$ and θ_i is not also algebraic over F_{i-1}.

Eg. $$\int \frac{e^{2x}f}{1+xe^x}\,dx = \int \frac{f\,\theta_1^2}{1+x\theta_1}\,dx.$$

$$F_1 = \mathbb{C}(x)(\theta_1 = e^x) \qquad e^{2x} = (e^x)^2$$

We will not use $F_2 = \mathbb{C}(x)(\theta_1 = e^{2x}, \theta_2 = e^x = \sqrt{\theta_1})$ since θ_2 is algebraic over F_1 since $p(z) = z^2 - \theta_1 \in F_1[z]$ has $p(\underline{\theta_2}) = \theta_2^2 - \theta_1 = 0$.

12.6. The Logarithmic extension subcase.

Let $F = K(x)(\theta_1, \ldots, \theta_n)$ be an elem. ext. of $K(x)$.
Let $f \in F(\theta)$ where $\theta = \log u$, $u \in F$, $\theta \notin F$, $\theta' \neq 0$.
Transcendental case: θ is not algebraic over F.

$$\int \underset{\underset{F(\theta)}{\cap}}{f(x)} = \int \frac{a(\theta)}{b(\theta)} \qquad \int \overset{\text{polynomial part}}{P} + \int \frac{R}{b} \leftarrow \deg R < \deg b. \text{ rational part.}$$

$a, b \in F[$
$\gcd(a,b) = .$

$a(\theta) \div b(\theta)$ in $F[\theta]$ $\quad \underline{a} = b\underline{P} + \underline{R}.$
$a/b = P + R/b.$

Claim $\int P$ <u>or</u> $\frac{R}{b}$ is not elementary Then $\int f$ is not elementary

$$\int \frac{R}{b} = \frac{A}{B} + \int \frac{C}{D} \quad \text{where } b = BD,\ A, B, C, D \in F[\theta] \text{ and } D \text{ is square-free.}$$

$$\underset{\substack{\uparrow \\ \text{Hermite}\checkmark \\ \text{Horrowitz ?}}}{} \qquad \underset{\substack{\uparrow \\ \text{PF} \quad \text{TR} \\ +\text{L.T.} \quad +\text{L.T.}}}{}$$

E.g. $\int \frac{\log^3 x + 1}{\log^2 x} = \int \log x + \int \frac{1}{\underline{\log^2 x}} = x\log x - x - \frac{x}{\log x} + \int \frac{1}{\log x}$

Hermite.

$\uparrow$ L.T. $\Rightarrow$ not elementary.

$F(\theta) = \underline{\mathbb{Q}(x)}(\log x)$

Output : $x \log x - x - \frac{x}{\log x} + \boxed{\int \frac{1}{\log x}}$

Hermite Reduction

$\int \frac{P}{Q}$ where $P, Q \in F[\theta]$ where $\theta = \log u$ and $\deg_\theta P < \deg_\theta Q$.

Let $Q = q_1 q_2^2 \cdots q_k^k$ be the S.F. factorization of Q in $F[\theta]$.

So $q_i \in F[\theta]$, $\gcd(q_i, \frac{dq_i}{d\theta}) = 1$ $\gcd(q_i, q_j) = 1$.

Let $T = Q/q_k^k = q_1 q_2^2 \cdots q_k^{k-1}$.

Solve $\sigma \cdot q_k' \cdot T + \tau \cdot q_k = P$ for $\sigma, \tau \in F[\theta]$ with $\sigma = 0$ or $\deg \sigma < \deg q_k$

$$\int \frac{P}{Q} = \int \frac{\sigma q_k'}{q_k^{k-1}} + \int \frac{\tau}{Q/q_k} = -\frac{\sigma/(k-1)}{q_k^{k-1}} + \int \frac{\tau + \sigma'/(k-1) \cdot T}{Q/q_k}.$$

E.g. $\int \frac{1}{\log^2 x} = \int \frac{1}{\underline{\theta^2}}$ $\quad P = 1$, $Q = q_1 \cdot q_2^2$ $\quad q_1 = 1$, $q_2 = \theta$ $= \log x$, $k = 2$ $\quad T = 1$ $\quad q_2' = \frac{1}{x}$

$F(\theta) = \mathbb{Q}(x)(\log x)$

Solve $\sigma \cdot \frac{1}{x} \cdot 1 + \tau \cdot \theta' = 1$ for $\sigma, \tau \in F[\theta]$ $\quad \sigma = x$ $\sigma' = 1$. $\tau = 0$.

$$\int \frac{1}{\log^2 x} = \frac{-x/(2-1)}{\log^1 x} + \int \frac{0 + 1 \cdot/(2-1) \cdot 1}{\log x}$$

$$= \frac{-x}{\log x} + \int \frac{1}{\log x}.$$

In computing $q_1 q_2^2 \cdots q_k^k$ we have $\gcd(q_k, \frac{dq_k}{d\theta}) = 1$ in $F[\theta]$.

But solving $\sigma q_k' T + \tau q_k = P$ we need $\gcd(q_k, \frac{dq_k}{dx}) = 1$.

Theorem 12.6 Let $a \in F[\theta]$ where $\theta = \log u$, $u \in F$, $\theta' \neq 0$

Theorem 12.6. Let $\underline{a} \in F[\theta]$ where $\theta = \log u$, $u \in F$, $\theta' \neq 0$ and θ is not algebraic over F, F is a diff. field, $\deg_\theta a = n > 0$, $\mathrm{lc}\ a = 1$.

If $\gcd(a, \frac{da}{d\theta}) = 1$ then $\gcd(a, \underset{\smile}{\overset{F[\theta].}{\frac{da}{dx}}}) = 1$.

Proof: NB: Th 12.2 $\Rightarrow$ $\frac{da}{dx} = a' \in F[\theta]$ so is well defined.

Let $a = \prod_{i=1}^{n} (\theta - \alpha_i)$ where $a(\alpha_i) = 0$. E.g. $a = \theta^2 - x = (\theta - \underset{\alpha_1}{\sqrt{x}})(\theta + \underset{\alpha_2}{\sqrt{x}})$

NB: $\gcd(a, \frac{da}{d\theta}) = 1$
$\Rightarrow$ a is square-free
$\Rightarrow$ $\alpha_i \neq \alpha_j$

$$\frac{da}{dx} = \sum_{i=1}^{n} \left(\underbrace{\frac{u'}{u} - \alpha_i'}_{=0?}\right) \cdot \prod_{j \neq i} (\theta - \alpha_j)$$

Claim $\frac{u'}{u} - \alpha_i' \neq 0$.

Suppose $\frac{u'}{u} - \alpha_i' = 0$

$\int \Rightarrow$ $\theta - \alpha_i = k$. a constant.

$\Rightarrow$ $\theta - k = \alpha_i$

$\Rightarrow$ $a(\theta - k) = \underline{a(\alpha_i)} = 0$

$\Rightarrow$ $p(\theta) = 0$ where $p(z) = a(\overset{\theta}{z} - \overset{\substack{K \subset F \\ \psi}}{k})$.

But $a \in F[\theta]$ so $p(z) \in F[z]$.

$\Rightarrow$ θ is algebraic over F. $\boxtimes$

$$\text{Let } g = \gcd(a, \tfrac{da}{dx}) = \gcd\left(\prod_{i=1}^{n} \theta - \alpha_i,\ \sum_{k=1}^{n} \left(\frac{u'}{u} - \alpha_k'\right) \prod_{j \neq k} (\theta - \alpha_j)\right)$$

$\downarrow$ $\swarrow k \neq i$ $\downarrow k = i$

Consider

$$g_i = \gcd\left(\theta - \alpha_i,\ (\theta - \alpha_i) \cdot \Delta + \left(\frac{u'}{u} - \alpha_i'\right) \cdot \prod_{j \neq i} (\theta - \alpha_j)\right).$$

$$\Rightarrow\ g_i = \gcd\left(\underline{\theta - \alpha_i},\ \left(\underset{\overset{\cap}{F}}{\frac{u'}{u}} - \underset{\neq 0}{\alpha_i^{\cancel{\prime}}}\right) \prod_{j \neq i} (\theta - \alpha_j)\right) \text{ in } F[\theta]$$

$\alpha_i \neq \alpha_j$ $\Rightarrow$ $g_i = 1$

$\Rightarrow$ $g = 1$.

Horrowitz's Method for $F[\theta]$, $\theta = \log u$.

$$\int \frac{P}{Q} = \frac{A}{B} + \int \frac{C}{D}$$ where $\deg_\theta P < \deg_\theta Q$ and D is square-free.

$$Q = q_1 q_2^2 \cdots q_k^k, \quad B = \gcd\left(Q, \frac{dQ}{d\theta}\right) = q_2 q_3^2 \cdots q_k^{k-1}, \quad D = Q/B = q_1 q_2 \cdots q_k.$$

Solve for $A, C \in F[\theta]$ with $\deg_\theta A < \deg_\theta B$ and $\deg_\theta C < \deg_\theta D$.

$$\int \frac{dx}{\log^2 x} = \int \frac{dx}{\theta^2} = \frac{A}{\theta^1} + \int \frac{C}{\underset{\uparrow}{\theta}}$$

$\deg_\theta A < 1 \Rightarrow A = a_0(x) \in \mathbb{Q}(x)$

$\deg_\theta C < 1 \Rightarrow C = c_0(x) \in \mathbb{Q}(x)$.

$$F(\theta) = \underbrace{\mathbb{Q}(x)}_{F}(\log x) \qquad \theta = \log x \qquad \theta' = \frac{1}{x}$$

$$\int \frac{dx}{\log^2 x} = \frac{a_0(x)}{\theta} + \int \frac{c_0(x)}{\theta} = \frac{-x}{\log x} + \int \frac{1}{\log x}$$

$$\overset{\prime}{\Longrightarrow} \quad \frac{1}{\theta^2} = \frac{a_0'(x)}{\theta} - \frac{1}{\theta^2} \cdot \frac{1}{x} \cdot a_0(x) + \frac{c_0(x)}{\theta}$$

$$\overset{\times\theta^2}{\Longrightarrow} \quad 1 = a_0'(x)\theta - \frac{1}{x} a_0(x) + c_0(x)\theta.$$

$$\left\{\begin{array}{lll} [\theta] & 0 & = a_0'(x) + c_0(x) \\ [\theta^0] & 1 & = -\frac{1}{x} a_0(x). \end{array}\right.$$

Solve for $a_0 \in \mathbb{Q}(x) = F$

$c_0 \in \mathbb{Q}(x)$.

$$\Rightarrow a_0 = -x$$

$$\Rightarrow 0 = -1 + c_0(x) \Rightarrow c_0(x) = 1.$$