Lec6B Multivariate Polynomials

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Multivariate Polynomial Rings 2.6

Let R be a ring and $a \in R[x_1, x_2, \ldots, x_n]$ a multivariate polynomial.
Let $a = \sum_{i=1}^{t} a_i x_1^{e_{1i}} \cdot x_2^{e_{2i}} \cdot \ldots \cdot x_n^{e_{ni}}$ where $a_i \in R$.

Def. The vectors $[e_{1i}, e_{2i}, \ldots, e_{ni}]$ are called <u>exponent vectors</u>.

E.g. $a = 7y^4 + 3x^2y^2 + 2xy^4 - 7xy^2 - 2x^3$ in $\mathbb{Z}[x,y]$

$$\begin{array}{ccccc} 7y^4 & 3x^2y^2 & 2xy^4 & 7xy^2 & 2x^3 \\ [0,4] & [2,2] & [1,4] & [1,2] & [3,0] \\ 4 & 4 & 5 & 3 & 3 \end{array}$$

Def The <u>total degree</u> $\deg(a) = \max_{i=1}^{t} \sum_{j=1}^{n} e_{ji}$.
If R is an integral domain then $\deg(ab) = \deg(a) + \deg(b)$

The terms of $a \in R[x_1, x_2, \ldots, x_n]$ are usually ordered in some descending order on the exponent vectors.

<u>Pure lexicographical order.</u> If u and v are two exponent vectors then $u > v$ if $u_k > v_k$ and $u_j = v_j$ for $1 \le j < k$.

E.g. $a = -2x^3 + 3x^2y^2 + 2xy^4 - 7xy^2 + 7y^4$

$[3,0] > [2,2] > [1,4] > [1,2] > [0,4]$

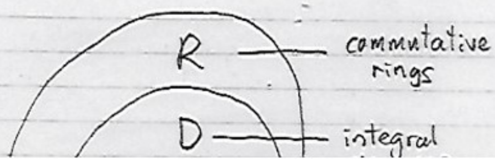
<u>Graded lexicographical order</u> (Maple 17,18) Terms are ordered in decreasing total degree with ties broken using pure lex. order.

E.g. $a = 2xy^4 + 3x^2y^2 + 7y^4 + 2x^3 - 7xy^2$

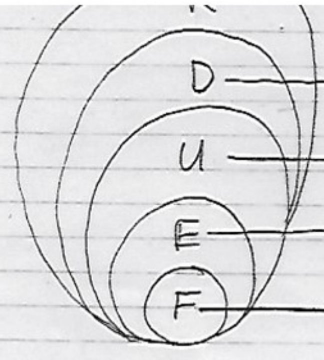
deg: $5 > 4 \quad 4 > 3 \quad 3$

For <u>both orderings</u> lc, lm, lt generalize in the obvious way.
E.g. for $\mathrm{lc}(a) = 2$, $\mathrm{lm}(a) = xy^4$, $\mathrm{lt}(a) = 2xy^4$. Moreover

(i) $\mathrm{lt}(ab) = \mathrm{lt}(a)\,\mathrm{lt}(b)$
(ii) $\mathrm{lc}(ab) = \mathrm{lc}(a) \cdot \mathrm{lc}(b)$ provided R is an integral domain.
(iii) $\mathrm{lm}(ab) = \mathrm{lm}(a)\,\mathrm{lm}(b)$



	Th 2.5	Th 2.6
	$R[x]$	$R[x,y]$
	$D[x]$	$D[x,y]$



Rings	$R[x]$	$R[x,y]$
integral domains	$D[x]$	$D[x,y]$
UFD	$U[x]$, $E[x]$	$U[x,y]$, $E[x,y]$, $F[x,y]$
Euclidean domains	$F[x]$	—
Fields	—	—

Why is $\underline{\mathbb{Z}[x]}$ not a Euclidean domain?

Division in $D[x_1,\dots,x_n]$

Let D be an int. dom. and $A, B \in D[x_1,\dots,x_n]$ $B \neq 0$.

How do we test if $B \mid A$? (with 0 remainder i.e. $A = B \cdot Q$).

If $B \mid A$ how do we compute the quotient Q.

Example Does $\overset{B}{xy+1} \mid \overset{A}{2y^2x^3 + (y^2+2y)x^2 + (2y+2)x}$ in $\underline{\mathbb{Z}[x,y]}$

$$D[x_1,x_2,x_3] \cong D[x_2,x_3][x_1] \cong D[x_3][x_2][x_1].$$

E.g. $A = 2x^3y^2 + x^2y^2 + 2yx^2 + 2xy + 2x \in \mathbb{Z}[x,y]$

In $\mathbb{Z}[x][y]$ $A = (2x^3+x^2)y^2 + (2x^2+2x)y + (2x)\cdot y^0.$

In $\mathbb{Z}[y][x]$ $A = (2y^2)x^3 + (y^2+2y)x^2 + (2y+2)x$

$$lc(A) = 2y^2 \quad lm(A) = x^3 \quad deg(A) = 3.$$

If $B \mid A$ then $A = B \cdot Q$ for some $Q \in \mathbb{Z}[y][x]$.

In $\mathbb{Z}[y][x]$ $lc(A) = lc(BQ) \quad lc(B) \cdot lc(Q)$

$\Rightarrow lc(B) \mid lc(A)$ in $\mathbb{Z}[y]$. a $\div$ in one less variable.

$\uparrow$ do this division recursively.

In $\mathbb{Z}[y][x]$ $lm(A) = lm(BQ) = lm(B) \cdot lm(Q)$

In $\mathbb{Z}[y][x]$ $\quad lm(A) = lm(BQ) = lm(B)\cdot lm(Q)$
$\Longrightarrow lm(B) \mid lm(A).$

$$\underbrace{2yx^2}_{q_1} + \underbrace{yx}_{q_2} \leftarrow Q$$

$B = (y)x + 1$

$lc(B)$ $lm(B)$

$(2y^2)x^3 + (y^2+2y)x^2 + (2y+2)x = A$

$lc(A)$ $lm(A)$

$R \leftarrow A$

$-[(2y^2)x^3 + (2y)x^2]$

$B \,)\, 0 + (y^2)x^2 + (2y+2)x \leftarrow R$

$-\ (y^2x^2 + yx) = q_2 \cdot B$

$= B \,)\, 0 + (y+2 \cdot x \leftarrow R.$

remainder R.

$y \mid 2y^2$ in $\mathbb{Z}[y]$

do it recursively.

$y \mid y^2$

do it recursively

$y \mid y+2$ in $\mathbb{Z}[y]$

do it recursively.

doesn't divide.

Stop because $y \nmid y+2$.

We have $A = B \cdot Q + (y+2)x$ and $B \nmid (y+2)\cdot x$

Hence $B \nmid A$. Algorithm stops if $lc(B) \nmid lc(R)$.
or $lm(B) \nmid lm(R)$.

If $B \mid A$ with quotient Q it won't matter
if we do the division in $\mathbb{Z}[x][y]$ or $\mathbb{Z}[y][x]$.