A3 instructor quesitons

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Q1 Let S be the sample space for the 5 coin tosses.

So $S = \{ HHHHH, TTTTT, \ldots \text{ etc.} \}$

The outcomes are like binary strings of length 5. So $|S| = 2^5$.

Let A_k be $\{$ outcomes with $\underline{\underline{k}}$ heads where $0 \le k \le 5 \}$

The number of ways to get k heads (and $n-k$ tails) is $\binom{n}{k}$.

So $\Pr(X=k) = |A_k|/|S| = \binom{n}{k}/2^n \underset{n=5}{=} \binom{5}{k}/32$

$$E[X] = \sum_{k=0}^{5} k \frac{\binom{5}{k}}{32} = \frac{1}{32}\left[0\cdot\binom{5}{0} + 1\cdot\binom{5}{1} + 2\cdot\binom{5}{2} + 3\cdot\binom{5}{3} + 4\cdot\binom{5}{4} + 5\cdot\binom{5}{5}\right]$$

$$= \frac{1}{32}\left[0 + 5 + 2\cdot 10 + 3\cdot 10 + 4\cdot 5 + 5\cdot 1\right] = 80/32 = 10/4 = \underline{\underline{2.5}}$$

Q2 See below.

Q3 Toss $n=5$ balls into $m=5$ bins randomly.

$\Pr(\text{bin } i \text{ has } 0 \text{ balls}) = \left(\frac{n-1}{n}\right)^m = \left(\frac{4}{5}\right)^5$

$\Pr(\text{bin } i \text{ has } 1 \text{ ball}) = n\left(\frac{1}{n}\right)\left(\frac{n-1}{n}\right)^{m-1} = \left(\frac{4}{5}\right)^4$.

$\Pr(\text{bin } i \text{ has } k \text{ balls}) = \underbrace{\binom{n}{k}}_{\#\text{ of ways}} \underbrace{\left(\frac{1}{n}\right)^k}_{k \text{ in bin } i} \underbrace{\left(\frac{n-1}{n}\right)^{m-k}}_{m-k \text{ not in bin } i} = \binom{5}{k}\left(\frac{1}{5}\right)^k\left(\frac{4}{5}\right)^{5-k}$

Tabulating the probabilities for $k=0,1,2,3,4,5$ we get

k	0	1	2	3	4	5
$\Pr(X=k)$	$\left(\frac{4}{5}\right)^5$	$\left(\frac{4}{5}\right)^4$	$10\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^3$	$10\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^2$	$5\cdot\left(\frac{1}{5}\right)^4\left(\frac{4}{5}\right)$	$\left(\frac{1}{5}\right)^5$

$$E[X] = 1\cdot\left(\frac{4}{5}\right)^4 + 20\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^3 + 30\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^2 + 20\left(\frac{1}{5}\right)^4\left(\frac{4}{5}\right) + 5\cdot\left(\frac{1}{5}\right)^5$$

$$= \frac{5\cdot 4^4 + 20\cdot 4^3 + 30\cdot 4^2 + 20\cdot 4 + 5 = 3125}{5^5 = 3125} = 1. \;☺$$

Q4 $E[X] = n\left(1-\frac{1}{n}\right)^m = n\left(\frac{n-1}{n}\right)^m$

$n=10, m=5 \quad E[X] = 5.905$

$n=10, m=10 \quad E[X] = 3.487$

$n=10, m=20 \quad E[X] = 1.216.$

If we toss 20 balls into 10 bins, on average there will be one empty bin.

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Q5 $\Pr(\text{bin } i \text{ has one ball}) = m\left(\frac{1}{n}\right)\left(\frac{n-1}{n}\right)^{m-1} = \frac{m}{n}\left(1-\frac{1}{n}\right)^{m-1}$.

Let Y be the # of bins with one ball.

Let $Y_i = \begin{cases} 1 & \text{if bin } i \text{ has one ball} \\ 0 & \text{otherwise.} \end{cases}$

$$E[Y] = E[Y_1+Y_2+\dots+Y_n] = nE[Y_i]$$
$$= n\cdot(Y_i\cdot\Pr(Y_i))$$
$$= n\cdot 1\cdot\left(\frac{m}{n}\right)\left(1-\frac{1}{n}\right)^{m-1} = \boxed{m\cdot\left(1-\frac{1}{n}\right)^{m-1}}$$
$$= m\left(\underbrace{\left(1-\frac{1}{n}\right)^{n}}_{e \text{ for large } n.}\right)^{(m-1)/n} \approx \boxed{m e^{-m/n}} \text{ for large } m.$$

For $m=n=10$ $\Pr(Y_i) = \frac{m}{n}\left(1-\frac{1}{n}\right)^{m-1} = \left(1-\frac{1}{10}\right)^{9} = 0.38742$.

$E[X] = n\cdot\Pr(Y_i) = m\left(1-\frac{1}{n}\right)^{m-1} = 10\left(1-\frac{1}{10}\right)^{9} = 3.8742$.

$E[X] \approx me^{-m/n} = 10e^{-1} = 3.6789$.

Q6 $a_{n+1} = 3a_n - a_{n-1}$ with $a_1 = 1, a_2 = 1$.

$a_3 = 3a_2 - a_1 = 3\cdot1-1 = 2$

$a_4 = 3a_3 - a_2 = 3\cdot2-1 = 5$

$a_2 = 3\cdot a_1 - a_0 \Rightarrow a_0 = 3a_1 - a_2 = 3\cdot1-1 = 2$.

Q7. $a_n - a_{n-1} = n$.

$a_n - a_{n-1} - a_{n-2} = 0$

Q8. Let i_n be the # people infected with COVID after n weeks.

We are given $i_n = 3i_{n-1}$ and $i_0 = 2.5$ million.

The general solution is $i_n = A\cdot 3^n$ for $n \geq 1$.

$i_0 = A = 2.5$ million

So $i_n = 2.5\ 3^n$ million.

We need to solve $i_n = 100 = 2.5\cdot 3^n$

$\Rightarrow 3^n = 40 \Rightarrow n = 3.358$ weeks.

Q9. The # comparisons in line 7 is $n-1$ because the loop is executed from 1 to $n-1$. The # comparisons in line 10 is C_{n-1} because the array to be sorted has length $n-1$. Thus For $n=1$, $C_1 = 0$ because of line 4. So

$C_n = (n-1) + C_{n-1}$ and $C_1 = 0$.

$\Rightarrow C_n = (n-1)+(n-2)+C_{n-2} = \cdots = (n-1)+(n-2)+\cdots+1+C_1$
$= (n-1)+(n-2)+\cdots+1+0 = \frac{n(n-1)}{2}$.

This exactly the same # comparisons as Bubblesort.

Q10. Let I, F, and B denote the Italian, French and Brazillian restaurants. For one evening we have $d_1 = 3$ choices.

For two evenings we have II, IF, IB, FI, FB, BI, BF, BB
so all 6 choices except FF. So $d_2 = 8$.

For three evenings we can't have _FF or FF_ which rules out 3+2 choices out of $3^3 = 27$ so $d_3 = 22$.

We can eat $\underset{1}{\underline{I}}\ \underset{2}{\underline{\quad}}\ \cdots\ \underset{n}{\underline{\quad}}$ so d_{n-1} choices
evening

I or B or F

Similarly if we go to B for the first evening. For F

$\underset{1}{\underline{F}}\ \underset{2}{\underline{\quad}}\ \cdots\ \underset{n}{\underline{\quad}}$ so $2d_{n-2}$ choices.

I or B

Thus $d_n = 2d_{n-1} + 2d_{n-2}$

$d_3 = 2d_2 + 2d_1 = 2\cdot 8 + 2\cdot 3 = 22$
$d_4 = 2\cdot d_3 + 2\cdot d_2 = 2\cdot 22 + 2\cdot 8 = 44 + 16 = 60$
$d_5 = 2\cdot 60 + 2\cdot 22 = 164$.

Q11

$a_n = \cancel{a_{n-1}} + An + B$
$\cancel{a_{n-1}} = \cancel{a_{n-2}} + A(n-1) + B$
$\cancel{a_{n-2}} = \cancel{a_{n-3}} + A(n-2) + B$
$\vdots$
$\cancel{a_2} = \cancel{a_1} + A(2) + B$
$\cancel{a_1} = 1$

Adding both sides of these equations we get
$a_n = 1 + A(n+n-1+\cdots+2) + (n-1)B = \left[\frac{n(n+1)}{2} - 1\right]A + (n-1)B + 1$.

Q12 $x_n = x_{n-1} + 20x_{n-2}$ with $x_0 = 5$, $x_1 = -2$.

$\Rightarrow x_n - x_{n-1} - 20x_{n-2} = 0$
$\Rightarrow r^2 - r - 20 = (r-5)(r+4) = 0 \Rightarrow r = 5, -4$.
$\Rightarrow x_n = A\cdot 5^n + B(-4)^n$

$$\left.\begin{array}{l} x_0 = A + B = 5 \\ x_1 = 5A - 4B = -2 \end{array}\right\} \Rightarrow 0 - 9B = -27 \Rightarrow B = 3, A = 2.$$

So $x_n = 2\cdot 5^n + 3(-4)^n$.

Q2 (b). The Theorem (Linearity of Expectation) says:
If X and Y are random variables and $a \in \mathbb{R}$ then
(1) $E(aX) = a \cdot E(X)$ and
(2) $E(X+Y) = E(X) + E(Y)$.
So if $Z = X - Y$ then

$$\begin{aligned} E(Z) &= E(X-Y) = E(X+(-Y)) \\ &= E(X) + E(-Y) \quad \text{by (2)} \\ &= E(X) + (-1)E(Y) \quad \text{by (1)} \\ &= E(X) - E(Y) = 7 - 3 = 4. \end{aligned}$$

Q2 (a) $E[aX] = \sum_{s \in S} aX(s) \cdot Pr(s) = a \left(\sum_{s \in S} X(s) Pr(s) \right) = a \cdot E[X]$.

a is a constant