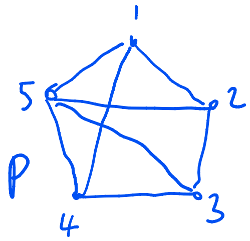
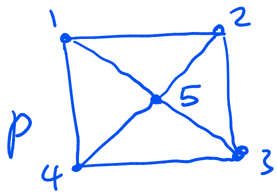
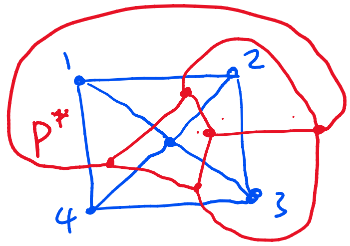
A7S21 Instructor Solutions

November 11, 2020 8:54 PM

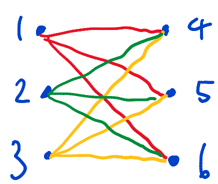
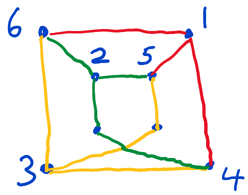
H.C. = Hamiltonian Cycle
H.P. = Hamiltonian Path

Q1

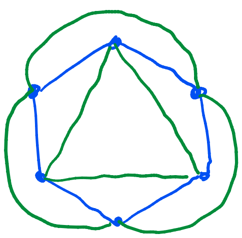
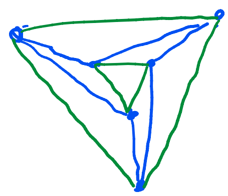




$P^* = P$ note.

Q2

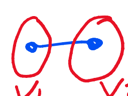
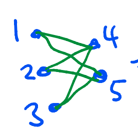
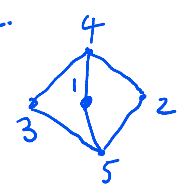



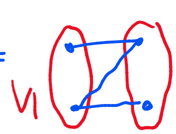
Q3 We need to find a planar drawing of a graph with 6 vertices and 12 edges. It will be 4-regular.

This drawing has straight lines.

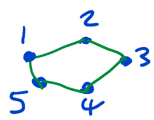
Q4 Let $G=(V,E)$ be a connected bipartite graph with $V = V_1 \cup V_2$.
Prove or disprove

(a) $|V_1| = |V_2|$ and $|V_1| \geq 2$ are necessary conditions for G to have a H.C.
TRUE (show if G does not meet either condition it may not have an HC.)
If $|V_1| < 2$ then G could be  which doesn't have a H.C.
If $|V_1| \neq |V_2|$ and $|V_1| \geq 2$ then G could be $K_{3,2}$ =  = 
which we showed in class does not have a H.C.

(b) $|V_1| = |V_2|$ and $|V_1| \geq 2$ are sufficient conditions for G to have a H.C.
FALSE.
Consider G =  . It satisfies G is connected, $|V_1| = |V_2| \geq 2$ but does not have a H.C.

Q5 Theorem 11.8.
Let $G=(V,E)$ be a loop-free graph (a simple graph) with $|V| = n \geq 2$. If $\deg(x)+\deg(y) \geq n-1$ for all $x,y \in V$, $x \neq y$, then G has a Hamiltonian Path.

The converse is:
Let $G=(V,E)$ be a simple graph with $|V|=n\geq 2$. If G has a Hamiltonian path then $\deg(x)+\deg(y) \geq n-1$ for all $x,y\in V$, $x\neq y$.



Consider a cycle on 5 vertices.
Notice $\deg(x)+\deg(y) = 4$.
This satisfies the condition $\deg(x)+\deg(y) \geq n-1=4$.
Consider a cycle on 6 vertices. So $\deg(x)+\deg(y)=4$
but $4=\deg(x)+\deg(y) \not\geq n-1=5$.

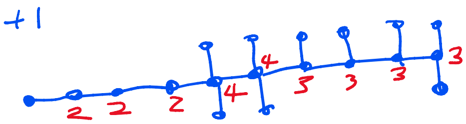
Q6. Let's construct a graph $G=(V,E)$ where the vertices are the students and an edge $\{u,v\}$ is in the graph if student u does NOT know student v. If each student knows at most 5 other students then each student does not know at least 6 students. So in G the $\deg(v)\geq 6$ and $n=|V|=12$. Theorem 11.8 says G has a Hamiltonian Path since $\underbrace{\deg x}_{\geq 6}+\underbrace{\deg y}_{\geq 6} \geq n-1=11$ for all pairs of vertices x,y in G.
The Hamiltonian Path is the seating arrangement.

Q7 We need to use $|V|=|E|+1$ for a tree and also $\sum_{v\in V} \deg(v) = 2|E|$. Suppose the tree has n leaf vertices. They all have degree 1. We are given

# vertices	degree
n	1
3	2
4	3
2	4

From this information
$$|E| = (n+3\times 2+4\times 3+2\times 4)/2$$
$$= (n+6+12+8)/2 = 13+n/2$$
$$|V| = n+3+4+2 = n+9$$

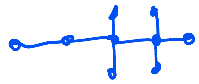
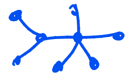
Now $|V|=|E|+1 \Rightarrow n+9 = 13+n/2+1$
$\Rightarrow n/2 = 5$
$\Rightarrow n=10$



Q8. Let $T=(V,E)$ be a tree. Use Euler's theorem

Q8. Let $T=(V,E)$ be a tree. Use Euler's theorem

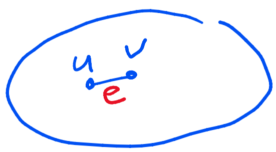
$$|V|-|E|+|F|=2$$

to show $|V|=|E|+1$. In a planar embedding of a tree
e.g.  or  there is only one face,
the infinite face, therefore

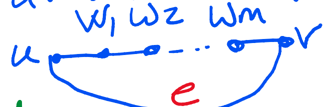
$$|V|-|E|+|F|=|V|-|E|+1=2 \Rightarrow |V|=|E|+1.$$

Q9. Let T be a tree with ≥ 2 vertices.
Prove that removing any edge in T disconnects T.

Let e be any edge in T. Let e be incident to vertices u and v in T so that T looks like

T 

Proof ① In class we proved there is a unique path between u and v, which is the path $P = u \overset{e}{\text{—}} v$. Therefore removing e leaves no path from u to v so T is disconnected.

Proof ② Let's remove e. Then $T-e$ is either connected or disconnected. Suppose $T-e$ is connected.
Then there is a path $u - w_1 - w_2 - \cdots - w_m - v$ in $T-e$. But then T has a cycle  which contradicts T is a tree.

Therefore T is disconnected.