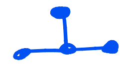
A2S21 Textbook problems

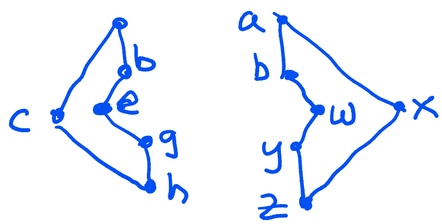
January 16, 2021 4:20 PM

11.1 Exercise 10. Give an example of a connected graph G where removing any edge results in a disconnected (unconnected) graph.

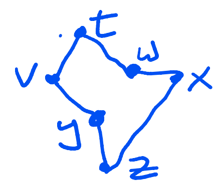
The smallest example is 

Any tree will do e.g.  or 

11.2 (a). Both graphs have 4 vertices of degree 2 and 4 vertices of degree 3. I notices that the first one G_1 has two cycles of length 6 (see below) but the second G_2 has only one so the two graphs are not isomorphic.



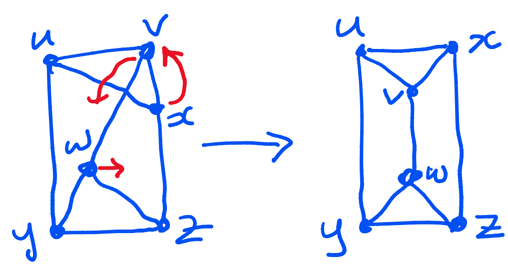
In G_1

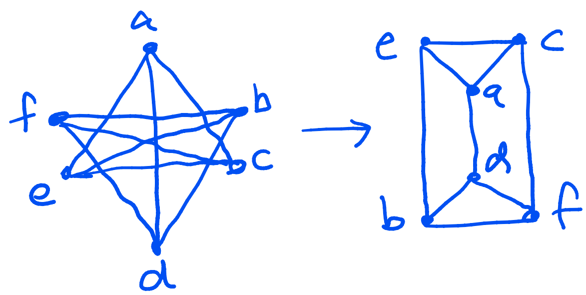


In G_2

(b). The second subgraph can be redrawn as a prism. The first one can too. So they are isomorphic. The isomorphism $f(x)$ is

x	a	b	c	d	e	f
$f(x)$	v	y	x	w	u	z





3.4 Exercise 7. Two integers are selected from $\{1,2,\ldots,100\}$ without replacement. What is the probability their sum is even?

Let $S = \{$ all pairs of integers from $\{1,2,\ldots,100\}\}$

Let $A = \{$ the pairs in S whose sum is even $\}$
$= \{(1,3),(1,5),\ldots,(2,4),(2,6),\ldots\}$.

We have $|S| = \binom{n}{2}$

$= \{(1,3),(1,5),\dots,(2,4),(2,6),\dots\}.$

We have $|S| = \binom{n}{2}$.

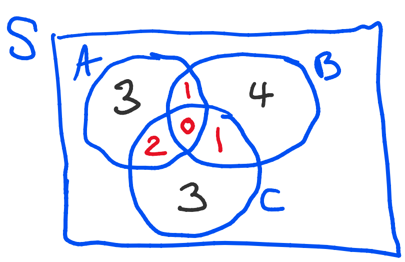
Since odd + odd = even and even + even = even

$A = \{\text{(odd, odd) pairs}\} \cup \{\text{(even, even) pairs}\}$

$|A| = \binom{50}{2} + \binom{50}{2}$. So $Pr(A) = \frac{|A|}{|S|} = \frac{\binom{50}{2}+\binom{50}{2}}{\binom{100}{2}} = \frac{49}{99}$.

two odd or two even

3.4 Exercise 12. One should draw a Venn diagram.



We are given $|A| = |B| = |C| = 6$

$|A \cap B| = |B \cap C| = 1$, $|A \cap C| = 2$, $|A \cap B \cap C| = 0$.

So we can fill in red.

Then we can fill in black.

a) How many models have exactly one feature? $3+4+3 = 10$.

b) How many models have none of the features?

$|S| - |A \cup B \cup C| = 15 - (3+1+4+2+0+1+3) = 15 - 14 = 1.$

c) If a model is selected at random what is the probability that it has exactly two features.

Let $X = \{\text{servers with 2 features}\}$

$|X| = 2+1+1 = 4$

$Pr(X) = \frac{|X|}{|S|} = \frac{4}{15}$

3.5 Exercise 2. Ashley tosses a coin 8 times.

(a) What is the probability she gets 6 heads e.g. HHTHTHH.

Let $S = \{\text{set of all outcomes}\}$.

Since each toss can result in heads H or tails T the number of outcomes $|S| = 2^8$. It's the same as the # of binary strings of length 8.

Let $A = \{\text{outcomes with 6 heads}\}$ $|A| = \binom{8}{6} = 28$

Let $A = \{$ outcomes with 6 heads$\}$ $\quad |A| = \binom{8}{6} = 28$

$$\Pr(A) = \frac{|A|}{|S|} = \frac{\binom{8}{6}}{2^8} = \frac{28}{2^8} = \frac{7}{64}$$

(b) What is the probability she gets at least 6 heads.

Let $B = \{$ outcomes with 7 heads$\}$ $\quad |B| = \binom{8}{7} = \binom{8}{1} = 8$

Let $C = \{$ outcomes with 8 heads$\}$ $\quad |C| = \binom{8}{8} = 1$

$\Pr($ she gets ≥ 6 heads$)$

$= \Pr($she gets 6 or 7 or 8 heads$)$

$= \Pr(A) + \Pr(B) + \Pr(C)$

$$= \frac{|A|}{|S|} + \frac{|B|}{|S|} + \frac{|C|}{|S|} = \frac{28}{2^8} + \frac{8}{2^8} + \frac{1}{2^8} = \frac{37}{256}.$$

(c). What is the probability she gets 2 heads?

Getting 2 heads means getting 6 ~~tails~~ tails.

The probability of getting 6 tails equals the probability of getting 6 heads so $7/64$

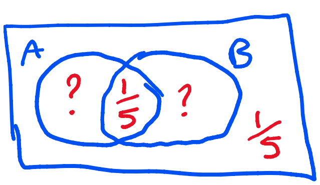
(d) What is the probability of getting at most 2 heads?

Getting ≤ 2 heads means getting ≥ 6 tails.

This probability is the same as part (b) so $\frac{37}{256}$.

3.5 Excercise 12.

Given $\Pr(A) = \Pr(B)$ and $\Pr(A \cap B) = \frac{1}{5}$ and $\Pr(\overline{A \cup B}) = 1/5$



$\Pr(A \cup B) = 1 - \Pr(\overline{A \cup B}) = 1 - \frac{1}{5} = 4/5$

$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) = 2 \cdot \Pr(A) - \Pr(A \cap B)$.

Thus $\Pr(A) = \frac{\Pr(A \cup B) + \Pr(A \cap B)}{2} = \frac{(4/5 + 1/5)}{2} = \frac{1}{2}$.

Thus $Pr(A) = \frac{Pr(A \cup B) + Pr(A \cap B)}{2} = \frac{(4/5 + 1/5)}{2} = \frac{1}{2}$.

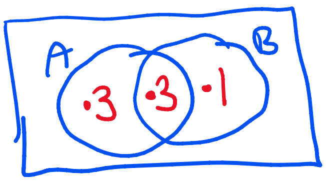
$Pr(A - B) = Pr(A) - Pr(A \cap B) = \frac{1}{2} - \frac{1}{5} = 3/10$.

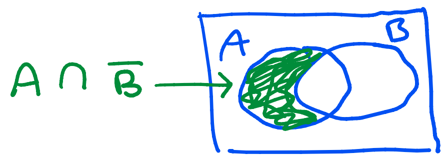
3.6 Exercise 2

Given $Pr(A) = 0.6$, $Pr(B) = 0.4$, $Pr(A \cup B) = 0.7$.

$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$

$\Rightarrow Pr(A \cap B) = Pr(A) + Pr(B) - Pr(A \cup B) = 0.6 + 0.4 - 0.7 = 0.3$





$Pr(A|B) = Pr(A \cap B)/Pr(B) = 0.3/0.4 = 0.75$

$Pr(A|\overline{B}) = Pr(A \cap \overline{B})/Pr(\overline{B})$

$= [Pr(A) - Pr(A \cap B)]/(1 - Pr(B))$

$= (.6 - .3)/(1 - 0.4) = 0.3/0.6 = 0.5$

3.6 Exercise 8

A coin is loaded so that $Pr(H) = 2/3$ and $Pr(T) = 1/3$.

Todd tosses the coin twice.

Let A be the event "the first toss is a tail"

So $A = \{TT, TH\}$

Let B be the event "both tosses are the same"

So $B = \{HH, TT\}$

Are A and B independent?

We need to check if $Pr(A \cap B) = Pr(A) \cdot Pr(B)$.

$Pr(A \cap B) = Pr(TT) = 1/3 \cdot \frac{1}{3} = \frac{1}{9}$

$Pr(A) = Pr(TT) + Pr(TH) = \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} = \frac{3}{9}$

$Pr(B) = Pr(HH) + Pr(TT) = \frac{2}{3} \cdot \frac{2}{3} + \frac{1}{3} \cdot \frac{1}{3} = \frac{5}{9}$.

We have $Pr(A \cap B) = \frac{1}{9} \neq \frac{3}{9} \cdot \frac{5}{9} = Pr(A) \cdot Pr(B)$

so the events are <u>not</u> independent