A1S21 Instructors Solutions

January 14, 2021 11:24 AM

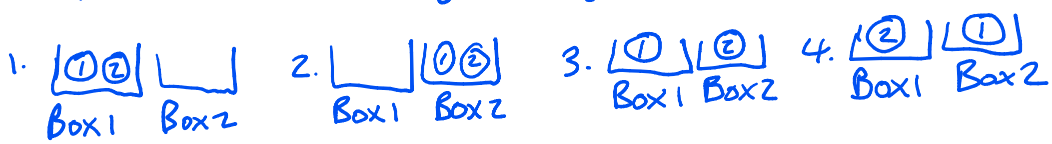
Q1. $\binom{5}{3} = \frac{5!}{3!\,2!} = \frac{5 \cdot 4}{2} = 10$ so there are 10 binary strings with three 1 bits.

11100 01110 00111 10110 10011 11010 11001 01110 01011 00111

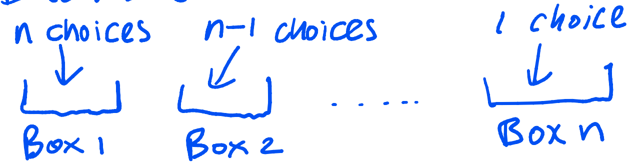
One needs a strategy — an ordering — to generate them.

Q2 (a). Each ball can go into each bin with no restriction.

So there are n^n ways. E.g. for $n=2$ there are $2^2=4$ ways.



(b). Each box must have one ball.



This is $n!$ It's the permutations of the n balls.

Q3 Let $\Sigma = \{A, B, C, D\}$ How many palindromes of length n?

Case $n=2$: there are 4 palindromes: AA, BB, CC, DD.

Case $n=3$: we can have AyA, ByB, CyC, DyD for any y

So there are 16.

Case $n=4$: we can have $xyyx$ for any $x \in \Sigma$ and $y \in \Sigma$

So there are 16 4 choices 4 choices

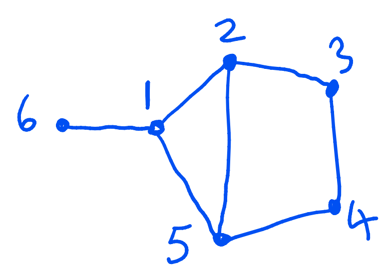
Case $n=2m$: we have $x_1 x_2 \cdots x_m x_m \cdots x_2 x_1$ for $x_i \in \Sigma$

So 4^m choices

Case $n=2m+1$: we have $x_1 x_2 \cdots x_m y x_m \cdots x_2 x_1$ for $x_i, y \in \Sigma$

So 4^{m+1} choices

Q4 Since vertex 6 is only adjacent to vertex 1:



A path of length 5 has 5 edges so 6 vertices.

Also a path must have distinct vertices; it can't have a cycle.

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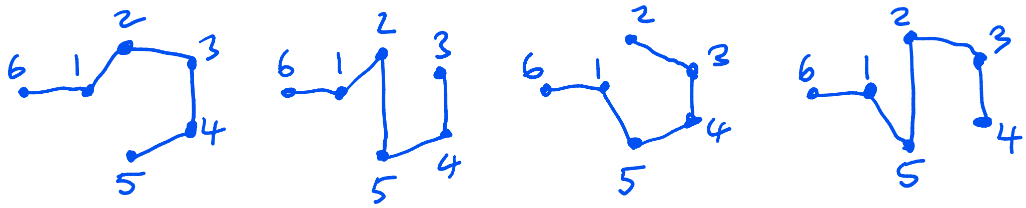
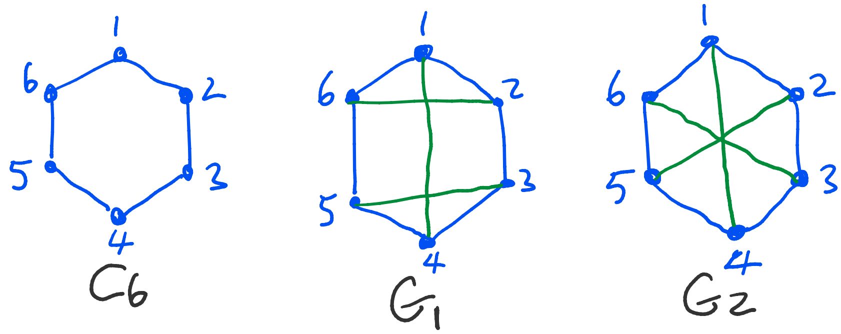
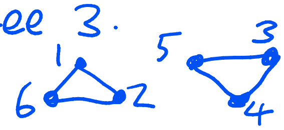


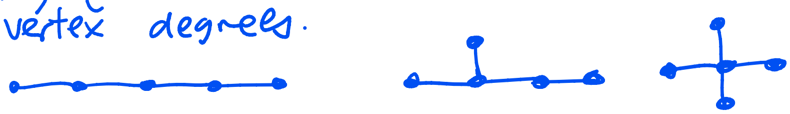
These are distinct because they have different edges.

Q5 Start with C_6 the cycle graph on 6 vertices.



All vertices in G_1 and G_2 have degree 3.
$G_1 \neq G_2$ because G_1 has triangles  in it
but G_2 has no triangles.

Q6 Draw all ("different") (unlabelled) trees with 5 vertices.
I will list the vertex degrees.



Vertex degrees 1 2 2 2 1   1 3 2 1   1 1 1 1 4.

These are different because the vertex degrees differ.
Notice that the sum of the vertex degrees = 8.

Q7. See class notes

Q8. Show that $\binom{n}{2} + \binom{n-1}{2}$ is a perfect square.

$-3^2.$

$$\binom{n}{2} = \frac{n!}{2!\,(n-2)!} = \frac{n\cdot(n-1)\cdot(n-2)!}{2!\,(n-2)!} = \frac{n(n-1)}{2}$$

$$\binom{n-1}{2} = \frac{(n-1)!}{2!\,(n-3)!} = \frac{(n-1)(n-2)(n-3)!}{2!\cdot(n-3)!} = \frac{(n-1)(n-2)}{2}$$

$$\binom{n}{2} + \binom{n-1}{2} = (n-1)\left[\frac{n}{2} + \frac{n-2}{2}\right] = (n-1)(n-1).$$

Q9 (a) How many ways can there be 6 balls of the same colour?

$$\binom{10}{6}\cdot 4$$

ways of choosing 6 from 10. ; 4 colours

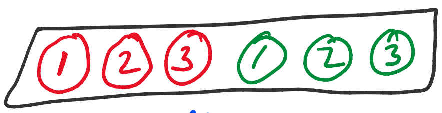
(b) 4 of one colour and 2 of another colour?

$$4\cdot\binom{10}{4}\cdot 3\cdot\binom{10}{2}$$

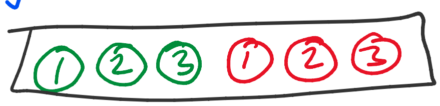
first colour ; second colour

(c) 3 of one colour and 3 of a different colour?

$4\cdot\binom{10}{3}\cdot 3\cdot\binom{10}{3}$ is wrong because it counts



and



as different choices, but they are the same.
We need to choose "pairs" of colours so

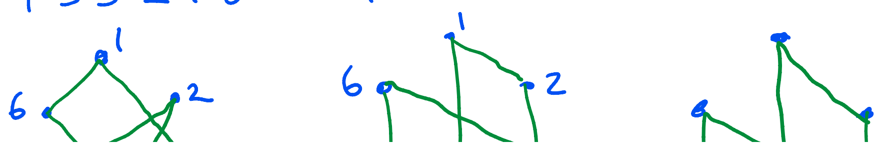
$$\underbrace{\binom{4}{2}}_{\text{Pair of colours}}\binom{10}{3}\binom{10}{3} = 6\cdot\binom{10}{3}\cdot\binom{10}{3}$$

(d) Two of one colour, two of another colour, 2 of a third colour.
We have 4 colours. We need to choose 3 colours. So

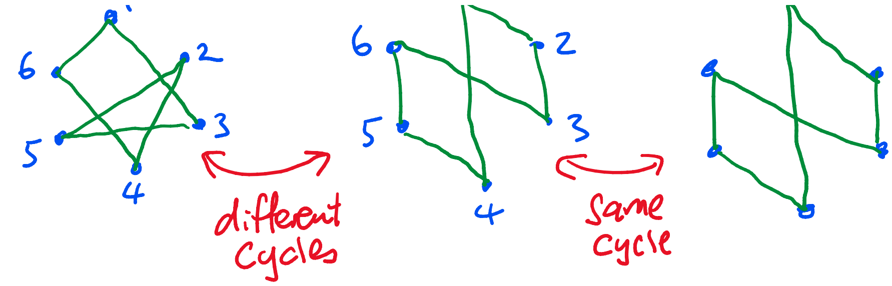
$$\underbrace{\binom{4}{3}}_{\text{Colours}}\underbrace{\binom{10}{2}\binom{10}{2}\binom{10}{2}}_{\text{2 of each colour}}.$$

Q10 (a) How many cycles on 6 vertices are there in K_6?
Since K_6 has all possible edges, each permutation of 123456 gives a cycle, for example.

Permutation 135246 123654 236541
Cycle



Cycle



There are $6!$ permutations of the vertices $1,2,3,4,5,6$ but some of the cycles are the same.

If a permutation is reversed the cycles would be the same

e.g. $123456 \to 654321$

If a permutation is rotated e.g.

$123456 \to 234561 \to 345612 \to 456123 \to 561234 \to 612345$

these are the same cycles. For each cycle there are 6 rotations and 2 reverses so 12 permutations for each cyle. So there are

$$\frac{6!}{12} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{12} = 60 \text{ cycles of length 6 in } K_5.$$

Q 10 (b). How many cycles of length 5 (edges) are in K_6?

Every choice of 5 vertices from $V = \{1,2,3,4,5,6\}$ gives us $\frac{5!}{5\cdot 2}$ different cycles. So

rotations reversals

$$\text{So } \binom{6}{5}\cdot\frac{5!}{5\cdot2} = 6\cdot\frac{5\cdot4\cdot3\cdot2}{5\cdot2} = 6\cdot12 = \underline{\underline{72}} \text{ cycles.}$$

Q 10 (c). How many cycles of length 4 (edges) are in K_6?

$$\binom{6}{4}\cdot\frac{4!}{4\cdot2} = \frac{6\cdot5}{2}\cdot\frac{24}{8} = 15\cdot3 = \underline{\underline{45}}$$

choose vertices rotations reversals

Q 10 (d) How many cycles of length 3 (triangles) are in K_6?

There is only one cycle on 3 vertices

$$\binom{6}{3}\cdot\frac{3!}{3\cdot2} = \frac{6\cdot5\cdot4}{3\cdot2\cdot1}\cdot 1 = \underline{\underline{20}}$$

$$\binom{6}{3} \cdot \frac{3!}{3\cdot 2} = \frac{6\cdot 5\cdot 4}{3\cdot 2\cdot 1} \cdot 1 = \underline{\underline{20}}$$

Choose vertices

1 cycle on 3 vertices

Q11 Given $f = (x+y+z)^7$ and $g = (x+y-3z)^7$.

(a) Coefficient of $x^2y^3z^2$ in

$$f = x^7 + \cdots + \binom{7}{2,3,2} x^2y^3z^2 + \cdots$$

$$g = x^7 + \cdots + \binom{7}{2,3,2} x^2y^3(-3z)^2 + \cdots$$

Coefficient

$$\frac{7!}{2!\,3!\,2!} = \frac{7\cdot 6\cdot 5\cdot 4}{2\cdot 2} = 210$$

$$\frac{7!}{2!\,3!\,2!}(-3)^2 = 210\cdot 9 = 1890$$

(b) What is the sum of the coefficients of f and g?

The sum of the coefficients of $f = (x+y+z)^7$ is obtained by substituting $x=y=z=1$ into f so $= (1+1+1)^7 = 3^7$

The sum of the coefficients of $g = (x+y-3z)$ is $(1+1-3\cdot 1)^7 = (-1)^7 = -1$.

(c) How many terms are in the expansion of f and g.

In the expansion of $f = (x+y+z)^7$ the terms are of the form

$cx^iy^jz^k$ for some coefficient c and $\boxed{i+j+k=7}$

The number of terms is the number of solutions to

$i+j+k=7$ with $i \geq 0$, $j \geq 0$ and $k \geq 0$

This is $\binom{7+3-1}{7} = \binom{9}{7} = \frac{9\cdot 8}{2} = \underline{\underline{36}}$.

The number of terms in the expansion of g is also $\underline{\underline{36}}$.

Q12 (a). How many ways can Mary distribute 12 apples and 10 oranges to her 3 children?

Apples: Let x_i be the number of apples child i gets. Then

$x_1 + x_2 + x_3 = 12$ where $x_i \geq 0$.

This has $\binom{12+3-1}{12} = \binom{14}{12}$ integer solutions so $\binom{14}{12}$ ways.

Oranges: Let y_i be the # oranges that child i gets. Then

Oranges: Let y_i be the # oranges that child i gets. Then

$$y_1+y_2+y_3=10 \text{ where } y_i \geq 0.$$

This has $\binom{10+3-1}{10} = \binom{12}{10}$ integer solutions so $\binom{12}{10}$ ways.

By the rule of product there are $\binom{14}{12}\cdot\binom{12}{10}$ different ways.

Q12(b). So 12 apples and 10 oranges but each child gets at least two apples and one orange. From part (a)

Apples: $x_1+x_2+x_3=12$ with $x_i \geq 2$

$\Rightarrow a_1+a_2+a_3=6$ with $a_i \geq 0$

This has $\binom{6+3-1}{6} = \binom{8}{6} = \frac{8\cdot 7}{2} = 28$ solutions.

Oranges: $y_1+y_2+y_3=10$ with $y_i \geq 1$

$\Rightarrow O_1+O_2+O_3=7$ with $O_i \geq 1$

This has $\binom{7+3-1}{7} = \binom{9}{7} = \frac{9\cdot 8}{2} = 36$ solutions.

By the rule of product there are $28\cdot 36$ ways.