Due Date: Monday February 8, 2021, 11:00 pm

Instructions

Answer all questions on paper or a tablet using your own handwriting. Please number each page. Include a cover page with your name, student ID number and a list of the questions you answered. If you use paper make a photo of each page and upload your solutions to crowdmark. If you use a tablet, export your assignment to .pdf and upload the .pdf to crowdmark.

Textbook Reading

- Section: 3.7 – we will not cover variance
- Sections: 10.1

Definitions, Concepts & Keywords

- Random variables, expected value
- The bins and balls problem and the coupon collectors problem
- Construct and solve first order recurrence relations

Exercises

A. **Textbook Questions**

Section 3.7 Exercise 2.
Section 10.1 Exercises 1, 2.

B. **Instructors Questions**

1. Suppose John tosses a coin 5 times. Let X be the number of heads.
   (a) Calculate $Pr(X = x)$ for $x \in \{0, 1, 2, 3, 4, 5\}$ and find $E(X)$.
   (b) If Y is the number of tails, what is $E(Y)$?
2. Let X be a random variable with $r(X) = \{1, 2, 3, 4\}$ and $\Pr(X = x) = c/x$ for some constant c.
   (a) Determine c.
   (b) Calculate $\Pr(X = x)$ for $x \in \{1, 2, 3, 4\}$ and find $E(X)$.
3. In a card game each player writes down 4 distinct cards from a standard deck of 52 cards. The dealer draws one card at random from the deck. The player wins if they chose that card.
   (a) What is the expected number of times the player must play the game to win the game?
   (b) If the player must pay 1 dollar to play the game and gets 10 dollars if they win the game, what is the expected winnings if the player plays the game 20 times.
4. If X be binomially distributed with parameters p and n then

$$\Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \text{ for } 0 \le k \le n.$$

   Show that the probabilities add to 1, that is, show that

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1.$$

   Hint: it's easy if you use the binomial theorem!
5. Suppose we toss a coin 4 times. Let p be the probability of getting heads. Calculate the probability of getting 3 heads for $p = 1/2$ and $p = 2/3$.

6. (a) Let X be a random variable and a be a constant. Show that $E(aX) = aE(X)$.
   Follow the proof that $E(X+Y) = E(X) + E(Y)$ given in class.
   (b) Let X and Y be two random variables. Suppose we know $E(X) = 7$ and $E(Y) = 3$.
   Let $Z = X - Y$ be a random variable. What is $E(Z)$? Justify your answer.
7. Suppose we toss 3 balls into 3 bins randomly. Let X be the number of balls in a bin.
   (a) Determine $\Pr(X = x)$ for $x = 0, 1, 2, 3$.
   (b) Calculate $E(X)$. You should get 1.
8. Suppose we toss m balls into n bins randomly. Let X be the number of bins with exactly **one** ball in them. Determine a formula for the probability p that a bin has one ball in it and then calculate $E(X)$, the expected number of bins with one ball. For $m = n = 10$ calculate p and $E(X)$. You should get $p = 0.38742$.
9. Every summer David and Jennifer take a vacation at a resort in Florida. The resort gives each guest a free dinner at one of the 5 local restaurants. If the restaurant is chosen by the resort at random, how many times, on average, must David and Jennifer stay at the resort before they get a free dinner at all 5 restaurants ?
10. Consider the recurrence $a_{n+1} = 3a_n - a_{n-1}$ with $a_1 = 1$, $a_2 = 1$.
    (a) Calculate a_0, a_3, a_4.
    (b) Is this recurrence a first order recurrence or second order?
    (b) Is this recurrence homogeneous or non-homogeneous?
11. Below is C code, and Python code, for a crazy function. The input is an array A of n integers.

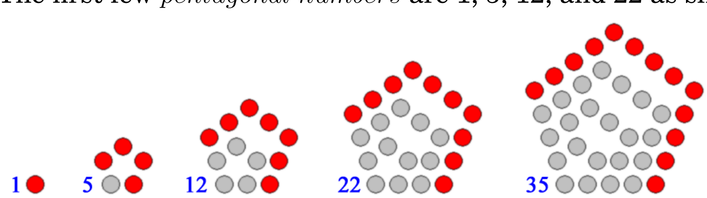
    C code:

```
1: int crazy( int A[], int n ) {
2:       int i,s;
3:       if( n<=1 ) return A[0];
4:       for( i=1; i<n; i++ )
5:           A[i-1] += A[i];
6:       s = crazy(A,n-1);
7:       return s;
8: }
```

    python code:

```
1:   def crazy(A,n):
2:       if n<=1:
3:           return A[0]
4:       for i in range(1,n):
5:           A[i-1] += A[i]
6:       s = crazy(A,n-1)
7:       return s
```

    To estimate how long crazy function takes to execute as a function of n, we will count the number of times line 5 is executed. Let c_n be the number of times line 5 is executed.
    (a) Give a recurrence equation and an initial value for c_n.
    (b) Solve the recurrence for c_n and determine c_{10} and c_{100}.
12. As I write this the number of people in Canada currently with COVID is 56 thousand. Government officials are very worried about the new strain of the virus from the UK because it is estimated to be 1.4 times more contagious than the dominant strain in Canada. Suppose it takes hold in Canada and the number of people with COVID increases by a factor of 1.2 every week.
    (a) Let i_n be the number of people with COVID after n weeks. Give a recurrence relation for i_n and solve the recurrence using $i_0 = 56$ thousand.
    (b) How many weeks will it take before the number of people with COVID increases by a factor of 10 to 560 thousand active cases?
13. Solve the following recurrences for the given initial values
    (a) $a_n = a_{n-1} + 2n$ for $n > 1$ and $a_1 = 1$.
    (b) $a_n = a_{n-1} + (n-1)^2$ for $n > 1$ and $a_1 = 0$.
14. The first few *pentagonal numbers* are 1, 5, 12, and 22 as shown in the figure below.



Find and solve a recurrence relation for the nth pentagonal number.

15. Consider P_n the path graph on n vertices. Suppose vertices can be coloured red, green or blue. Let c_n be the number of ways the vertices in P_n can be coloured such that no two adjacent vertices are coloured blue.
    (a) How many colourings are there for P_1, P_2, and P_3.
    (b) Find a (second order) recurrence for c_n.
    (c) Using the recurrence calculate c_3, c_4 and c_5.