Due Date: Monday Februaru 8, 2021, 11:00 pm

Instructions

Answer all questions on paper or a tablet using your own handwriting. Please number each page. Include a cover page with your name, student ID number and a list of the questions you answered. If you use paper make a photo of each page and upload your solutions to crowdmark. If you use a tablet, export your assignment to .pdf and upload the .pdf to crowdmark.

Textbook Reading

- Section: 3.7 – we will not cover variance
- Sections: 10.1

Definitions, Concepts & Keywords

- Random variables, expected value, the bins and balls problem.
- Construct and solve first order recurrence relations.

Exercises

A. **Textbook Questions**

Section 3.7 Exercise 2.

Section 10.1 Exercises 1, 2.

B. **Instructors Questions**

1. Suppose John tosses a coin 5 times. Let X be the number of heads.
   Calculate $Pr(X = x)$ for $x = 0, 1, 2, 3, 4, 5$ and find $E(X)$.
2. (a) Let X be a random variable and a be a constant. Show that $E(aX) = aE(X)$.
   Follow the proof that $E(X + Y) = E(X) + E(Y)$ given in class.

   (b) Let X and Y be two random variables. Suppose we know $E(X) = 7$ and $E(Y) = 3$.
   Let $Z = X - Y$ be a random variable. What is $E(Z)$? Justify your answer.
3. Suppose we toss 5 balls into 5 bins randomly. Let X be the number of balls in a bin.

   (a) Determine the probability X is 0, 1, 2, 3, 4, 5 balls.

   (b) Calculate $E(X)$. You should get 1.
4. Suppose we toss m balls into n bins randomly.
   Let X be the expected number of empty bins.
   In class we showed that $E(X) = n(1 - 1/n)^m$. For $n = 10$ and $m = 5, 10, 20$ calculate $E(X)$.
   For the case $n = 10$ and $m = 20$, interpret the result for $E(X)$ (write one sentence in English to explain what the result means).
5. Suppose we toss m balls into n bins randomly. Let X be the number of bins with exactly **one** ball in them. Determine a formula for the probability p that a bin has one ball in it and then calculate $E(X)$, the expected number of bins with one ball. For $m = n = 10$ calculate p and $E(X)$. You should get $p = 0.38742$.
6. Consider the recurrence $a_{n+1} = 3a_n - a_{n-1}$ with $a_1 = 1$, $a_2 = 1$. Calculate a_0, a_3, a_4.
7. Give an example of a first order non-homogenous recurrence relation and a second order homogeneous recurrence relation.
8. As I write this the number of people in India with the corona virus is 2.5 million. It is estimated that each person who has the virus will, on average, infect 3 people in a week before recovering or dying. Let i_n be the number of people infected after n weeks where $i_0 = 2.5$ million. Give a recurrence relation for i_n. How long will it take before 100 million are infected?

9. Below is C code, and Python code, for the Insertionsort algorithm. The input is an array A of n decimal numbers. Line 10 sorts the first $n-1$ numbers in A recursively.

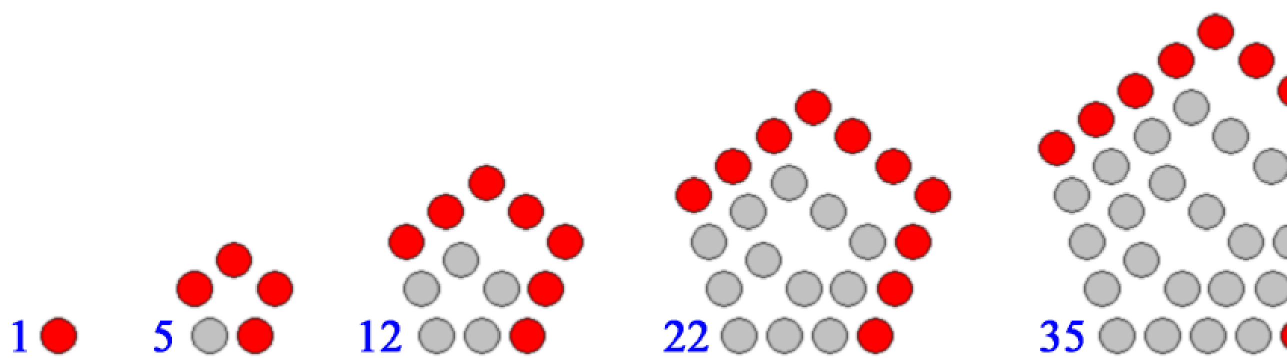
   C Code:
```
 1: void Insertionsort( double A[], int n ) {
 2:     // sort A[0],A[1],...,A[n-1] into ascending order
 3:     int i,m; double t;
 4:     if( n<=1 ) return; // A is already sorted
 5:     Insertionsort(A,n-1);
 6:     t = A[n-1];
 7:     for( i=n-2; i>=0 && A[i]>t; i-- )
 8:         A[i+1] = A[i];
 9:     A[i+1] = t;
10:    }
```

   Python Code:
```
 1: def Insertionsort(A,n):
 2:     # sort A[0], A[1], ..., A[n-1] into ascending order
 3:     if n<=1:
 4:         return # A is already sorted
 5:     InsertionSort(A,n-1)
 6:     t = A[n-1];
 7:     i = n-2
 8:     while i>=0 and A[i]>t:
 9:            A[i+1] = A[i]
10:             i -= 1
11:     A[i+1] = t
```
   Let c_n be the number of comparisons between elements of A made in line 7. Give a recurrence equation and an initial value for c_n and solve it. Does Selectionsort do fewer or more comparisons than Bubblesort?
   Note: you do not need to know how an algorithm works in order to count the number of times it executes a particular operation.

10. Consider P_n the path graph on n vertices. Suppose vertices can be coloured red, green or blue. Let c_n be the number of ways the vertices in P_n can be coloured such that no two adjacent vertices are coloured blue.
    (a) How many colourings are there for P_1, P_2, and P_3.
    (b) Find a recurrence for c_n.
    (c) Using the recurrence calculate c_3, c_4 and c_5.

11. Solve the following recurrences for the given initial values
    (a) $a(n) = a(n-1) + 2n \quad$ for $\quad n > 1 \qquad a(1) = 1$
    (b) $a(n) = a(n-1) + (n-1)^2 \quad$ for $\quad n > 1 \qquad a(1) = 0$

12. The first few *pentagol numbers* are 1, 5, 12, and 22.



Find and solve a recurrence relation for the nth pentagonal number

13. Suppose there are n different types of coupons in a bin. Assume there are equal number of each coupon. If we draw coupons at random from the bin, how many draws will it take to get at least one of each coupon?
14. In a card game each player writes down 4 distinct cards from a standard deck of 52 cards. The dealer draws one card at random from the deck. The player wins if they chose the card that the dealer drew.
    (a) What is the expected number of times the player must play the game to win the game? Use the geometric distribution.
    (b) If the player must pay 1 dollar to play the game and gets 10 dollars if they win the game, what is the expected winnings if the player plays the game 20 times.
15. For any probability distibution, the sum of the probabilities must be 1. Let T be geometrically distributed with parameter p. In class we showed that

$$\Pr(T = k) = p(1-p)^{k-1} \text{for} k \geq 1$$

and $E(T) = 1/p$. Show that the probabilities sum to 1.