Solution AS521 Instructor Q's.

1) 238 804.

2) a) $[x^{15}]\left(x+x^3+x^5+\dots+x^{15}\right)^3$

Using Wolfram we get 28

If $a_1+a_2+a_3=k$ where k is even and a_i odd, then the answer is 0 because the sum of three odd numbers is odd.

b) $[x^{16}]\left(x^2+x^3+x^4+x^{10}+x^{11}+\dots x^{16}\right)\left(1+x^5+x^{10}+x^{15}\right)\left(1+x^2+x^4+\dots x^{16}\right)$

(using Wolfram) we get 10.

3) We expand $(x+x^2+\dots x^6)^5$ and we see the max coefficient are at 17 and 18.

4) $[x^{60}]\left(1+x^2+\dots x^{60}\right)\cdot\left(1+x^5+x^{10}+\dots x^{60}\right)\left(1+x^8+x^{16}+\dots x^{64}\right)$

By Wolfram α, we get 23.

5) from $C(x)=1+x+x^2+\dots=\frac{1}{1-x}$

$$C'(x)=1+2x+3x^2+\dots=1/(1-x)^2$$

$$xC'(x)=x+2x^2+3x^3+\dots=x/(1-x)^2$$

$$A(x)=xC'(x)-x=2x^2+3x^3+\dots=x/(1-x)^2-x$$

$$=\frac{x-x(1-x)^2}{(1-x)^2}=\frac{x-x(1-2x+x^2)}{(1-x)^2}$$

$$=\frac{2x^2-x^3}{(1-x)^2}$$

$$B(x) = \frac{1}{1-3x}$$

$$\therefore \quad A(x)+B(x) = \frac{2x^2-x^3}{(1-x)^2} + \frac{1}{(1-3x)}$$

$$= \frac{(2x^2-x^3)(1-3x) + (1-x)^2}{(1-x)^2(1-3x)}$$

$$= \frac{2x^2-6x^3-x^3+3x^4+1-2x+x^2}{(1-2x+x^2)(1-3x)}$$

$$= \frac{1-2x+3x^2-7x^3+3x^4}{1-3x-2x+6x^2+x^2-3x^3}$$

$$= \frac{1-2x+3x^2-7x^3+3x^4}{1-5x+7x^2-3x^3}$$

$$A(x) = \frac{2x^2-x^3}{(1-x)^2} = \frac{-x+2x^2-x^3+x}{(1-x)^2} = \frac{-x(1-x)^2}{(1-x)^2} + \frac{x}{(1-x)^2} = -x + \frac{x}{(1-x)^2}$$

$$\therefore A'(x) = -1 + \frac{1(1-x)^2 - x\,2(1-x)(-1)}{(1-x)^4}$$

$$= -1 + \frac{(1-x)+2x}{(1-x)^3} = \frac{-(1-x)^3 + 1+x}{(1-x)^3}$$

$$= \frac{x^3-3x^2+3x-1+1+x}{(1-x)^3}$$

$$= \frac{x^3-3x^2+4x}{1-3x+3x^2-x^3}$$

6) using the formula

$$\frac{1}{(1-x)^k} = \sum_{n\geq 0} \binom{n+k-1}{n} x^n$$

we have

$$\frac{1}{(1-3x)^2} = \sum_{n\geq 0} \binom{n+1}{n} (3x)^n = \sum_{n\geq 0} (n+1)\,3^n x^n \quad \therefore c_n = 3^n(n+1)$$

7) $a_n = [x^n]\underbrace{(1+x+x^2+\dots)^2}_{\text{red \& yellow balls}}(1+x^2+x^4+\dots)$

$= [x^n]\ \frac{1}{(1-x)^2}\cdot\frac{1}{1-x^2} = [x^n]\ \frac{1}{(1-x)^2(1-x^2)}$

8) a) 2 to 12

b) Let die$_1$ have sides 1,3,4,5,6,8 and die$_2$ 1,2,2,3,3,4.

Number	Rolls (1st die + 2nd die)	total was
2	1+1	1
3	1+2, 1+2. ← not a typo (die$_2$ has 2 2's)	2
4	1+3, 1+3, 3+1	3
5	1+4, 3+2, 3+2 + 4+1	4
6	3+3, 3+3, 4+2, 4+2, 5+1	5
7	3+4, 4+3, 4+3, 5+2, 5+2, 6+1	6
8	4+4, 5+3, 5+3, 6+2, 6+2	5
9	5+4, 6+3, 6+3, 8+1	4
10	6+4, 8+2, 8+2	3
11	8+3, 8+3	2
12	8+4	1

$\therefore\ P(x) = x^2 + 2x^3 + 3x^4 + 4x^5 + 5x^6 + 6x^7 + 5x^8 + 4x^9 + 3x^{10} + 2x^{11} + x^{12}$

c) $D_1(x) = x + x^3 + x^4 + x^5 + x^6 + x^8$ $\quad D_2(x) = x + 2x^2 + 2x^3 + x^4$

d) Check $D(x) = D_1(x)\cdot D_2(x)$

e) Same!

a) $[x^n] \frac{x}{(3+x)^2} = [x^{n-1}] \frac{1}{(3+x)^2}$

$= [x^{n-1}] \frac{1}{9(1+\frac{x}{3})^2}$

$= \frac{1}{9}[x^{n-1}] \frac{1}{(1+\frac{x}{3})^2} = \frac{1}{9}[x^{n-1}] \sum_{i \geq 0} (i+1)\left(\frac{-x}{3}\right)^i$

$= \frac{1}{9}\left(\frac{-1}{3}\right)^{n-1} n$

$[x^n] \frac{x}{(2-x)^3} = [x^{n-1}] \frac{1}{(2-x)^3} = [x^{n-1}] \frac{1}{8} \frac{1}{(1-\frac{x}{2})^3}$

$= \frac{1}{8} [x^{n-1}] \sum_{i \geq 0} \binom{i+2}{i} \left(\frac{x}{2}\right)^i$

$= \frac{1}{8}\left(\frac{1}{2}\right)^{n-1} \binom{n+1}{n-1}$

$= \frac{1}{2^{n+2}} \binom{n+1}{2}$ $\quad \forall n \geq 0$

$[x^n] 5x^2 = \begin{cases} 5 & \text{if } n=2 \\ 0 & \text{if } n \neq 0 \end{cases}$

$\therefore [x^n] \frac{x}{(2-x)^3} + 5x^2 = \begin{cases} \frac{1}{2^{n+2}}\binom{n+1}{2} & \text{if } n \neq 2 \\ \frac{83}{16} & \text{if } n = 2 \end{cases}$

10) Set $f(x) = 0 + x - 2x^2 + 4x^3 - 8x^4 + 16x^5 + \dots$

$$= x(1 - 2x + 4x^2 - 8x^3 + 16x^4 + \dots)$$

$$= x \frac{1}{1+2x} = \frac{x}{1+2x}$$

11) $A(x) = \dfrac{1}{(3-x)(3+x)} = \dfrac{A}{3-x} + \dfrac{B}{3+x} \Rightarrow 1 = (3+x)A + B(3-x)$

$A = 1/6, \quad B = 1/6$

$$\therefore A(x) = \frac{1}{6(3-x)} + \frac{1}{6}\frac{1}{(3+x)}$$

$$= \frac{1}{18}\frac{1}{1-\frac{x}{3}} + \frac{1}{18}\frac{1}{1+\frac{x}{3}}$$

$$\therefore [x^n]A(x) = \frac{1}{18}[x^n]\frac{1}{1-\frac{x}{3}} + \frac{1}{1+\frac{x}{3}}$$

$$= \frac{1}{18}[x^n]\sum_{i\geq 0}\left(\frac{x}{3}\right)^i + \sum_{i\geq 0}\left(-\frac{x}{3}\right)^i$$

$$= \begin{cases} \frac{1}{3^{n+2}} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

12) $A(x) = \dfrac{1}{(x-2)^2(1-3x)} = \dfrac{A}{(x-2)^2} + \dfrac{B}{(x-2)} + \dfrac{C}{1-3x} \Rightarrow 1 = A(1-3x) + B(x-2)(1-3x) + C(x-2)^2$

$\Rightarrow A = -1/5, \quad C = 9/25,$

$\Rightarrow 1 = -2A + B(1-2)(1-3) + C$

$= +2/5 + 2B + 9/25$

$= 19/25 + 2B$

$\Rightarrow \frac{6}{25} = 2B \Rightarrow 3/25 = B$

$$B(x) = -\frac{1}{5}\frac{1}{(x-2)^2} + \frac{3}{25}\frac{1}{(x-2)} + \frac{9}{25}\frac{1}{1-3x}$$

$$B(x) = -\frac{1}{5}\frac{1}{(x-2)^2} + \frac{3}{25}\frac{1}{(x-2)} + \frac{9}{25}\frac{1}{1-3x}$$

$$\Rightarrow [x^n]\ B(x) = [x^n]\ -\frac{1}{20}\frac{1}{(1-\frac{x}{2})^2} - \frac{3}{50}\frac{1}{1-\frac{x}{2}} + \frac{9}{25}\frac{1}{1-3x}$$

$$= -\frac{1}{20}\sum_{i\geq 0}(n+1)\left(\frac{x}{2}\right)^i - \frac{3}{50}\sum_{i\geq 0}\left(\frac{x}{2}\right)^i + \frac{9}{25}\sum_{i\geq 0}(3x)^i$$

$$= -\frac{1}{20}(n+1)\frac{1}{2^n} - \frac{3}{50}\frac{1}{2^n} + \frac{9}{25}3^n$$

13) $A(x) = \frac{1}{1+x+x^2} \Rightarrow A(x)(1+x+x^2) = 1.$

$$\Rightarrow \sum_{n\geq 0} a_n x^n + x\sum_{n\geq 0} a_n x^n + x^2\sum_{n\geq 0} a_n x^n = 1$$

$$\Rightarrow \sum_{n\geq 0} a_n x^n + \sum_{n\geq 0} a_n x^{n+1} + \sum_{n\geq 0} a_n x^{n+2} = 1$$

$$\therefore \begin{array}{l} a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\ a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + \cdots \\ a_0 x^2 + a_1 x^3 + a_2 x^4 + \cdots \end{array} = 1$$

$\therefore a_0 = 1.$

$a_1 + a_0 = 0 \Rightarrow a_1 = -1$

$a_2 + a_1 + a_0 = 0 \Rightarrow a_2 = 0$

$a_3 + a_2 + a_1 = 0 \Rightarrow a_3 + 0 + -1 = 0$

$\Rightarrow a_3 = 1.$

$a_4 + a_3 + a_2 = 0 \Rightarrow a_4 + 1 + 0 = 0$

$\Rightarrow a_4 = -1.$

Reccurence: $a_n + a_{n-1} + a_{n-2} = 0 \quad \forall n \geq 2.$

(4) $A(x) = \dfrac{1}{1-x-x^2-x^3}$

$$(1-x-x^2-x^3)\,A(x) = 1 \Rightarrow \sum_{n\geq 0} a_n x^n - \sum_{n\geq 0} a_n x^{n+1} - \sum_{n\geq 0} a_n x^{n+2} - \sum_{n\geq 0} a_n x^{n+3} = 1$$

$$\Rightarrow a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$$
$$- (a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + \dots)$$
$$- (a_0 x^2 + a_1 x^3 + a_2 x^4 + \dots)$$
$$- (a_0 x^3 + a_1 x^4 + a_2 x^5 + \dots) = 1$$

$\therefore a_0 = 1$

$a_1 - a_0 = 0 \Rightarrow a_1 = 1$

$a_2 - a_1 - a_0 = 0 \Rightarrow a_2 - 1 - 1 = 0 \Rightarrow a_2 = 2$

$a_3 - a_2 - a_1 - a_0 = 0 \Rightarrow a_3 - 2 - 1 - 1 = 0$

$\Rightarrow a_3 = 4$

$a_4 - a_3 - a_2 - a_1 = 0 \quad a_4 - 4 - 2 - 1 = 0 \Rightarrow a_4 = 7$

$a_5 - a_4 - a_3 - a_2 = 0 \Rightarrow a_5 - 7 - 4 - 2 = 0 \Rightarrow a_5 = 13$

$a_n - a_{n-1} - a_{n-2} - a_{n-3} = 0 \qquad \forall n \geq 3.$