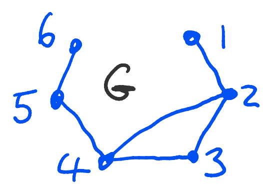
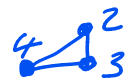
A2S21 Instructors' Problems Solutions

January 15, 2021 5:33 PM

Q1 In the code we see that $1 \le i \le n$, $1 \le j < i$ and $1 \le k < j$ or $1 \le k < j < i \le n$. So the first time the counter is set to 1 happens for $k=1, j=2, i=3$. The condition $k<j<i$ means the set $\{i,j,k\}$ always has 3 distinct integers. So counter equals the number of subsets of size 3 from $\{1,2,3,\dots,n\}$ which is $\binom{n}{3}$.

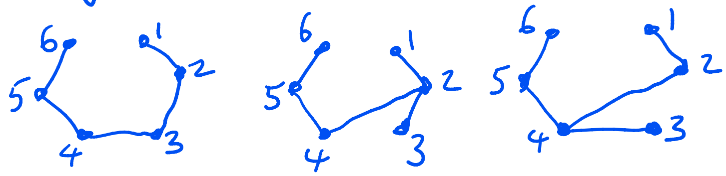
Q2



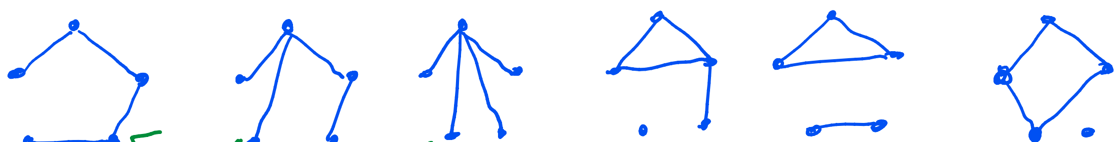
(b) G has a triangle


in it. So G cannot be bipartite.

(c) The spanning trees must include the edges $\{1,2\}$, $\{4,5\}$, $\{5,6\}$. So



Q3

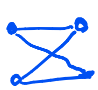
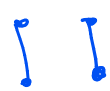


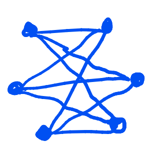
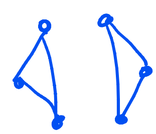
Degrees 11222 11123 11114 01223 11222 02222

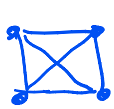
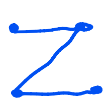
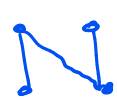
trees

Q4 (a) $K_{1,1}$ =

$\overline{K_{1,1}}$ =


$K_{2,2}$ =

$\overline{K_{2,2}}$ =


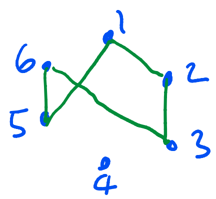
$K_{3,3}$ =

$\overline{K_{3,3}}$ =


(b) K_4 =

G =

$\overline{G}$ =

G and $\overline{G}$ are path graphs of length 3 edges.

(c) If G and $\overline{G}$ are isomorphic then they must have the same number of edges. Also if $K_n=(E,V)$ and $G=(E_1,V_1)$ and

(c) If G and $\overline{G}$ are isomorphic then they must have the same number of edges. Also if $K_n = (E, V)$ and $G = (E_1, V_1)$ and $\overline{G} = (E_2, V_2)$ then $E = E_1 \cup E_2$. Thus $|E|$ must be even. Notice that K_4 has 6 edges which is even. But in K_6, the number of edges $= \binom{6}{2} = \frac{6!}{4!2!} = \frac{6 \cdot 5}{2} = 15$. which is odd.

Q5. Let $3 \le k \le n$. How many cycles of length k are in K_n?
We saw on assignment 1, any permutation of the k vertices in K_6, e.g. 1 2 3 6 5 for $k=5$, corresponds to a cycle.

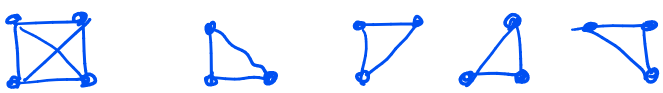


We also saw that there are k permutations (rotations) 23651, 36512, 65123, 51236, 12365 and, for each rotation, 2 permutations (reversals) 32156, 63215, 56321, 15632, 21563, which all correspond to the same cycle.

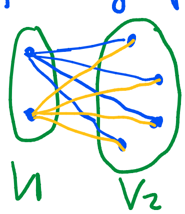
Thus for each choice of k vertices there are $k!/2k$ cycles.
Thus the total # cycles with k vertices in K_n is

$$\binom{n}{k} \cdot \frac{k!}{2k}.$$

Check: for $n=4, k=3$: $\binom{4}{3} \frac{3!}{2 \cdot 3} = 4$. Here are the 4 triangles:

K_4 

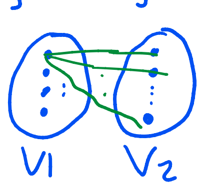
Q6 (a) How many subgraphs of K_n are isomorphic to $K_{2,4}$.

$K_{2,4} =$ 

We have to choose 2 vertices for V_1 and 4 different vertices for V_2.
So $\binom{n}{2} \cdot \binom{n-2}{4}$ subgraphs.

For $n=6$ this is $\binom{6}{2} \cdot \binom{4}{4} = 15$ which is all pairs of vertices for V_1.

(b) How many subgraphs of $K_{n,n}$ are isomorphic to $K_{2,4}$?

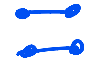
$K_{n,n} =$ 

We must choose 2 vertices from V_1 and 4 from V_2 OR 4 from V_1 and 2 from V_2.

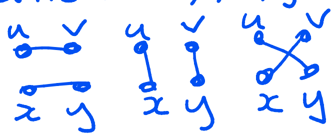
Thus the # subgraphs is $\binom{n}{2}\binom{n}{4} + \binom{n}{2}\binom{n}{4} = 2\binom{n}{2}\binom{n}{4}$

$\bar{V_1}$ $\bar{V_2}$ and 2 from V_2.

The # subgraphs is $\binom{n}{2}\cdot\binom{n}{4}+\binom{n}{4}\cdot\binom{n}{2}=2\binom{n}{2}\binom{n}{4}$.

Q7 (a). How many subgraphs of K_n are isomorphic to  G

One way is to choose 4 vertices u, v, x, y.

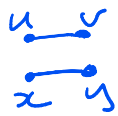
For this choice there are  subgraphs of K_4 isomorphic to G.

So the answer is $\binom{n}{4}\cdot 3$.

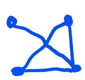
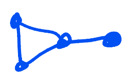
Another way is to first choose a pair of vertices $u, v \in V$.

There is only one way to choose one edge to get u — v

Repeat this for the second pair of vertices to get x — y.

But this chooses the subgraph  twice.

There are $\binom{n}{2}\cdot\binom{n-2}{2}/2$ subgraphs.

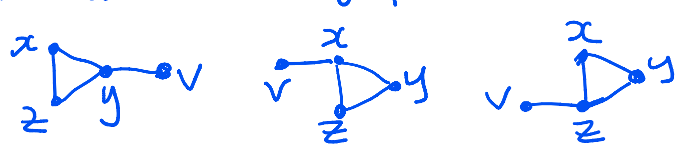
(b) Let $G=$  $=$ 

How many subgraphs of K_n are isomorphic to G?

There are $\binom{n}{3}$ ways to choose 3 vertices x, y, z for the triangle in G.

For this choice there is one way to choose 3 edges.

There is $\binom{n-3}{1}$ ways to choose the fourth vertex v. We can construct a graph isomorphic to G in 3 ways



So there are $\binom{n}{3}\cdot\binom{n-3}{1}\cdot 3 = 3\cdot(n-3)\binom{n}{3}$ subgraphs.

Q8 Let S be the set of ways of choosing 4 guests from 12.

So $|S|=\binom{12}{4}$.

(a). What is the probability that the host chooses 2 men & 2 women?

Let A = { choices with 2 men and 2 women }

[This is like a box of 6 numbered red balls and 6 numbered blue balls]

[This is like a box of 6 numbered red balls and 6 numbered blue balls]

$|A| = \underset{\text{women}}{\binom{6}{2}}\underset{\text{men}}{\binom{6}{2}}$. $\Pr(A) = \frac{|A|}{|S|} = \binom{6}{2}\binom{6}{2}/\binom{12}{4} = \frac{15 \cdot 15}{495} = \frac{5}{11}$

(b) What is the probability that the host chooses one man and 3 women?

Let $B = \{$ choices with one man and 3 women $\}$

$|B| = \binom{6}{3}\cdot\binom{6}{1}$. $\Pr(A) = \frac{|A|}{|S|} = \binom{6}{3}\cdot\binom{6}{1}/\binom{12}{4} = \frac{20\cdot 6}{495} = \frac{8}{33}$.

Q9 Let S be the set of all poker hands. $|S| = \binom{52}{5} = 2598960$.

(a). Let $f = \{$ all hands with 5 cards in the same suit $\}$

$|f| = \binom{13}{5}\cdot 4 = 5148$. $\Pr(f) = \frac{|f|}{|S|} = \frac{33}{16660} = 0.00198$.

(b) Let $p = \{$ all hands that have a pair $\}$

$$|p| = \underbrace{\binom{4}{2}\cdot 13}_{\#\text{ pairs}}\ \underbrace{\binom{12}{3}}_{3\text{ different kinds}}\cdot \underbrace{4^3}_{\text{colours}} = 1098240. \quad \Pr(p) = \frac{|p|}{|S|} = \frac{352}{833} = .44257$$

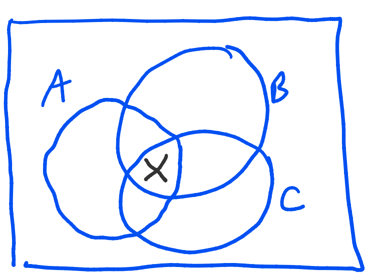
(c) Let $t = \{$ all poker hands with two pairs $\}$

$$|t| = \underbrace{\binom{4}{2}\cdot 13}_{\text{choose 1st pair}} \cdot \underbrace{\binom{4}{2}\cdot 12}_{\text{2nd pair}} \cdot \underbrace{44}_{\text{5th card}} \cdot \underbrace{\frac{1}{2}}_{\text{avoid double counting pairs}} = 123552$$

e.g. 7 7 J J 9
J J 7 7 9

$\Pr(t) = \frac{|t|}{|S|} = \frac{198}{4165} = .04754$

Q10



$X = A\cap B\cap C$

$\Pr(A\cup B\cup C) = \Pr(A) + \Pr(B) + \Pr(C) - \Pr(A\cap B) - \Pr(A\cap C) - \Pr(B\cap C)$

$\Pr(A)+\Pr(B)+\Pr(C)$ counts X 3 times
$-\Pr(A\cap B) - \Pr(A\cap C) - \Pr(B\cap C)$ subtracts X 3 times so we need to add it back

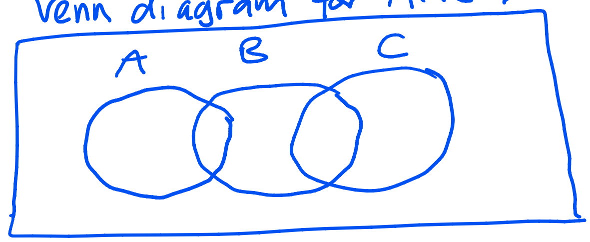
$+\Pr(A\cap B\cap C)$

Q11.
- A and B are independent $\Rightarrow \Pr(A\cap B) = \Pr(A)\cdot\Pr(B)$
- B and C are independent $\Rightarrow \Pr(B\cap C) = \Pr(B)\cdot\Pr(C)$
- A and C are disjoint $\Rightarrow A\cap C = \emptyset \Rightarrow \Pr(A\cap C) = 0$.

This also means $A\cap B\cap C = \emptyset$ so $\Pr(A\cap B\cap C) = 0$.

- A and C are disjoint $\Rightarrow$ $A \cap C = \emptyset$
  This also means $A \cap B \cap C = \emptyset$ so $Pr(A \cap B \cap C) = 0$.

Venn diagram for $A \cap C = \emptyset$



- $Pr(A) = 1/5$
- $Pr(C) = 2/5$
- $Pr(A \cup B \cup C) = 4/5$

$$Pr(A \cup B \cup C) = Pr(A) + Pr(B) + Pr(C) - Pr(A \cap B) - Pr(A \cap C) - Pr(B \cap C) + Pr(A \cap B \cap C)$$

$$\Rightarrow \quad 4/5 = 1/5 + Pr(B) + 2/5 - 1/5 \cdot Pr(B) - 0 - Pr(B) \cdot 2/5 + 0$$

$$\Rightarrow \quad 1/5 = (1 - 1/5 - 2/5) \cdot Pr(B) \Rightarrow Pr(B) = 1/5 / 2/5 = 1/2.$$

Q12 We are given Pr(had COVID and tests +ve) = 0.98
and Pr(has not had COVID and tests +ve) = 0.02

Let S be the set of all Americans
Let A be the Americans who've had COVID = 10%
Let B be the Americans who test +ve for COVID.

$$Pr(B) = \underbrace{.10 \times .98}_{\text{has COVID and tests +ve}} + \underbrace{.90 \times .02}_{\text{has not had COVID and tests +ve}} = .098 + .018 = .116.$$

The probability that an American who tests +ve has had COVID is

$$Pr(A|B) = \frac{Pr(B|A) \cdot Pr(A)}{Pr(B)} = \frac{0.98 \times 0.1}{.116} = 0.845.$$

This is quite low.