Lec25C Risch: exponential subcase: polynomial part
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Theorem 12.2 (differentiation of logarithmic polynomials)

Let F be a differential field and $F(\theta)$ be a logarithmic transcendental differential extension of F with $\theta' \neq 0$.
I.e. $\theta = \log(u)$ for some $u \in F$, $u' \neq 0$, $\theta \notin F$.

If $a = a_n\theta^n + \cdots + a_1\theta + a_0 \in F[\theta]$ with $n>0$, $a_n \neq 0$

(i) $a' = \frac{d}{dx} a(\theta) \in F[\theta]$
(ii) if $a_n' = 0$ then $\deg_\theta a' = n-1$
(iii) if $a_n' \neq 0$ then $\deg_\theta a' = n$

Theorem 12.3 (differentiation of exponential polynomials)

Let F be a differential field and $F(\theta)$ be an exponential transcendental differential extension of F with $\theta' \neq 0$.
I.e. $\theta = e^u$ for some $u \in F$, $u' \neq 0$, $\theta \notin F$

(i) If $a = a_n\theta^n + \cdots + a_1\theta + a_0 \in F[\theta]$ with $n>0$, $a_n \neq 0$

$$a' = \frac{d}{dx} a(\theta) \in F[\theta] \quad \text{and} \quad \deg_\theta a' = n$$

(ii) If $h \in F \setminus \{0\}$ and $m \in \mathbb{Z} \setminus \{0\}$ then

$$(h\theta^m)' = \bar{h}\theta^m \text{ for some } \bar{h} \in F\setminus\{0\}$$

(iii) $a(\theta) \mid a'(\theta) \Rightarrow a(\theta) = h\theta^m$ for some $h \in F$, $m \in \mathbb{Z}$

Example. $a = xe^x + \frac{1}{x}e^{2x} = x\theta + \frac{1}{x}\theta^2$
$F[\theta] = \mathbb{Q}(x)[e^x]$

$$a' = 1\cdot e^x + x\cdot e^x + -\frac{1}{x^2}e^{2x} + \frac{2}{x}e^{2x} =$$
$$= (1+x)\theta + \left(-\frac{1}{x^2} + \frac{2}{x}\right)\theta^2$$

$$\frac{d}{dx} \underset{\in F}{h}\theta^m = h'\cdot\theta^m + h\, m\theta^{m-1}\cdot\theta'$$
$$= h'\theta^m \; h\, m\theta^{m-1}\cdot w'\theta$$
$$= (h' + mhw')\theta^m$$

$(h' + mhw') = 0?$, $\in F$

$\theta = e^w$, $w \in F$
$\theta' = w'\theta$

If $h' + mhw' = 0 \Rightarrow (h\theta^m)' = 0$

$\Rightarrow h\theta^m = k$ for some constant k.

$\Rightarrow \theta^m - \underset{\substack{\wedge \\ F}}{k/h} = 0$

$\Rightarrow \theta$ is a root of $p(z) = z^m - k/h \in F[z]$

$\Rightarrow \theta$ is algebraic over F $\boxtimes$

12.7 Exponential Extension: Polynomial Part

Let $F = K(x)(\theta_1,\dots,\theta_n)$, $\theta = e^w$, $w \in F$, $w' \neq 0$, θ is NOT alegraic over F.

Let $\bar{P} = p_{-\ell}\theta^{-\ell} + \dots + p_0 + \dots + p_m\theta^m$ where $p_i \in F$, i.e., $\bar{P} \in F[\theta, \theta^{-1}]$.

By Liouville's Theorem, if $\int \bar{P}$ is elementary then

$$\int \bar{P}(\theta) = \underset{\in F(\theta)}{V_0(\theta)} + \sum c_i \log \underset{\in F(\theta)}{v_i(\theta)} \qquad \text{WLOG } v_i \in F[\theta],\ \theta \nmid v_i$$

$$= \bar{V}_0(\theta) + \frac{a(\theta)}{b(\theta)} + \sum c_i \log v_i(\theta) \qquad \bar{V}_0 \in F[\theta,\theta^{-1}],\ a,b \in F[\theta],\ \theta \nmid b$$

$\gcd(a,b) = 1$, $\gcd(v_i, \frac{dv_i}{d\theta}) = 1$

$\log \frac{A}{B} = \log A - \log B.$

$$\Rightarrow \underset{F[\theta,\theta^{-1}]}{\bar{P}(\theta)} = \underset{F[\theta,\theta^{-1}]}{\bar{V}_0(\theta)'} + \left(\frac{a(\theta)}{b(\theta)}\right)' + \sum c_i \frac{v_i(\theta)'}{v_i(\theta)} \quad \gcd = 1 \text{ by Th 12.8}$$

by Th 12.3

If $\deg_\theta b > 0$ The terms in the PFD of ( ) cannot cancel $\Rightarrow b \in F$.

If $\deg_\theta v_i > 0$ Then v_i^{-1} cannot cancel $\Rightarrow v_i \in F$.

$$\Rightarrow \int \bar{P}(\theta) = \bar{V}(\theta) + \sum c_i \log \underset{\in F}{v_i} \quad \text{where } \bar{V}(\theta) = \sum_{i=-k}^{j} \underset{\in F}{q_i}\theta^i$$

Since $(\sum L)' \in F$ and for $\theta = e^w$, $\theta' = w'\theta \neq 0$, for $i \neq 0$

$$(q_i\theta^i)' = q_i'\theta^i + q_i \cdot i w'\theta \cdot \theta^{i-1} = \underbrace{(q_i' + i q_i w')}_{\neq 0 \text{ by Th 12.3}}\theta^i$$

It follows That $j = m$ and $k = \ell$ i.e.,

$$\int p_{-\ell}\theta^{-\ell} + \dots + p_0 + \dots + p_m\theta^m = \underset{\in F}{q_{-\ell}}\theta^{-\ell} + \dots + \underset{\in F}{q_0} + \dots + \underset{\in F}{q_m}\theta^m + \sum c_i \log v_i$$

Example

$$\int xe^{-x} + x + \tfrac{1}{x} + e^{2x} = \int \underset{p_{-1}}{x}\theta^{-1} + \underset{p}{(x + \tfrac{1}{x})} + \underset{p_2}{\theta^2}$$

$F(\theta) = \mathbb{Q}(x)(e^x)$

$$= \underset{\in \mathbb{Q}(x)}{q_{-1}}\theta^{-1} + \tfrac{1}{2}x^2 + \underset{\in \mathbb{Q}(x)}{q_2}\theta^2 + \log x$$

$$\int p_{-\ell}\theta^{-\ell}+\cdots+p_0+\cdots+p_m\theta^m = q_{-\ell}\theta^{-\ell}+\cdots+q_0+\cdots+q_m\theta^m + \Sigma L$$

Equating coefficients in θ^j yields

$j\neq 0 \quad p_j = q_j' + j\omega' q_j \quad \Leftarrow \quad p_j\theta^j = [q_j\theta^j]'$ $\quad \theta = e^\omega \quad \theta' = \omega'\theta$

$j=0 \quad p_0 = q_0' + \Sigma L'$

The case $j=0$ $\int p_0$ where $p_0\in F$ is an $\int$ problem in F which can be solved recursively. Otherwise we must solve

$$\overset{F}{\overset{\Downarrow}{q_j'}} + j\overset{F}{\overset{\Downarrow}{\omega'}}\,\overset{F}{\overset{\Downarrow}{q_j}} = \overset{F}{\overset{\Downarrow}{p_j}} \quad \text{for } q_j\in F = K(x)(\theta_1,\ldots,\theta_n).$$

This is called a Risch differential equation.
If it has no solution in F then $\int p_j$ is not elementary.

Example $\int xe^{x^2}dx \qquad F(\theta) = \mathbb{Q}(x)(e^{x^2}) \qquad \omega = x^2$

$\omega' = 2x$

$\int x\theta\, dx = \overset{F}{\overset{\Downarrow}{q_1}}\theta + \underline{\text{constant}}$

$' \Rightarrow x\theta = q_1'\theta + q_1 2x\theta = (q_1' + 2x)\theta$

$\Rightarrow x = q_1' + 2xq_1$ and $q_1\in\mathbb{Q}(x)$

$\Rightarrow q_1 = \frac{1}{2}$ by inspection

$\int xe^{x^2}dx = q_1\theta = \frac{1}{2}e^{x^2}$

Example. $\int e^{-x^2}dx = \int \theta\, dx = q_1\theta + c. \quad \underline{\underline{q_1\in\mathbb{Q}(x)}}$

is not elementary.

polynomial in θ

$F(\theta) = \mathbb{Q}(x)[e^{-x^2}]$

$\theta = e^{-x^2}$

$\theta' = -2x\theta$

$\theta = q_1'\theta + q_1 \cdot -2x\theta$

$\Rightarrow \theta = (q_1' - 2xq_1)\theta.$

$\Rightarrow 1 = q_1' - 2xq_1$ for $q_1\in F = \mathbb{Q}(x).$

Let $q_1(x) = \frac{a(x)}{b(x)}$ with $a,b\in\mathbb{Q}[x]$ and $\gcd(a,b)=1$.

$$\Rightarrow 1 = \frac{a'}{b} - \frac{1\cdot a\cdot b'}{b^2} - 2x\frac{a}{b}$$

$$\times b^2 \quad \Rightarrow \quad b^2 = a'b - ab' - 2xab.$$

$$\Rightarrow \quad b \mid ab' \quad \underset{\gcd(a,b)=1}{\Rightarrow} \quad b \mid b' \quad \Rightarrow b' = 0$$
$$\Rightarrow b \in \mathbb{Q}$$

$\Rightarrow q_1 \in \mathbb{Q}[x]$ Let $q_1 = \underline{a_n x^n + \dots + a_0}$ where $\underline{n \geq 0}$. $a_n \neq 0$

$$1 = q_1' - 2x q_1$$

$$\Rightarrow \quad 1 = (n a_n x^{n-1} + \dots + a_1) - 2x(a_n x^n + \dots + a_0)$$

$[x^{n+1}]$ $\quad 0 = -2a_n \Rightarrow a_n = 0.$ $\boxtimes$

$\Rightarrow 1 = q_1' - 2x q_1$ has no solution for $q_1 \in \mathbb{Q}[x]$

$\Rightarrow \int e^{-x^2} dx$ is not elementary.

Exercise $\int \frac{e^x}{x} dx \longrightarrow$ not elementary.

Example. $\int \frac{1}{e^x} = \int \frac{1}{\theta} = q_{-1}\theta^{-1} + c.$

"polynomial" in θ

$F(\theta) = \mathbb{Q}(x)(\theta)$

$\theta = e^x$

$\theta' = e^x$

$$\frac{1}{\theta} = q_{-1}'\theta^{-1} - \theta^{-2}\theta' q_{-1}$$
$$= q_{-1}'\theta^{-1} - \theta^{-2}\theta \cdot q_{-1}$$
$$= (q_{-1}' - q_{-1})\theta^{-1}$$

$[\theta^{-1}]$ $\quad 1 = q_{-1}' - q_{-1}$ where $q_{-1} \in \mathbb{Q}(x)$.

Let $q_{-1} = \frac{a}{b}$ with $\gcd(a,b) = 1$ and $a, b \in \mathbb{Q}[x]$.

$$\Rightarrow \quad 1 = \frac{a'}{b} - \frac{b'a}{b^2} - \frac{a}{b}$$

$$\Rightarrow \quad b^2 = a'b - b'a - ab.$$

$$\Rightarrow \quad b \mid b'a \quad \Rightarrow \deg_x b = 0 \Rightarrow q_{-1} \in \mathbb{Q}[x].$$

Let $q_{-1} = a_n x^n + \dots + a_0$ where $n \geq 0$.

$$1 = q_{-1}' - q_{-1}$$

$$\Rightarrow \quad 1 = (n a_n x^{n-1} + \dots + a_1) - (\underline{a_n x^n} + \dots + a_0)$$

$$1 = q_{-1}' - q_{-1}$$

$$\Rightarrow 1 = (n a_n x^{n-1} + \dots + a_1) - (a_n x^n + \dots + a_0)$$

$$\Rightarrow n = 0.$$

$$1 = 0 - a_0 \Rightarrow a_0 = -1.$$

$$\Rightarrow q_{-1}(x) = -1.$$

$$\int \frac{1}{e^x} dx = q_{-1} \cdot \theta^{-1} = -1 \cdot e^{-x} + c.$$

$$1 \cdot e^{-x} = \frac{1}{e^x}.$$

"polynomial in $\theta = e^x$"

Exercise $\int \left(\log x\, e^x + \frac{1}{x} e^x + x e^{-x}\right) dx = \int \left(\left(\log x + \frac{1}{x}\right)\theta^1 + x\theta^{-1}\right) dx$

$$= q_1 \theta^1 + x q_{-1} \theta^{-1} + c.$$

where $q_1, q_{-1} \in F$.

$F(\theta) = \mathbb{Q}(x)(\log x)(e^x)$