Theorem 12.9 (Trager-Rothstein – Exponential case)

$\int \frac{C(\theta)}{D(\theta)}\,dx$ $\quad C, D \in F[\theta],\ \theta = e^{w},\ \theta \notin F,\ \theta' \neq 0.$
$0 \leq \deg_\theta C < \deg_\theta D,\ \mathrm{lc}_\theta D = 1,\ \theta \nmid D$
$\gcd(C, D) = 1,\ \gcd(D, D') = 1$ in $F[\theta]$

Let $R(z) = \mathrm{res}_\theta(\underbrace{C - zD'}_{\in F[\theta]}, D) \in F[z]$.

(i) $\int \frac{C}{D}\,dx$ is elementary iff roots of $R(z)$ are constants

(ii) If $\int \frac{C}{D}$ is elementary then $\int \frac{C}{D} = \sum c_i \log v_i + g_i$

where c_i are the distinct roots of $R(z)$
$v_i = \gcd(C - c_i D', D) \in F[\theta]$
$g_i = -c_i \deg_\theta(v_i) \cdot w$

Example $\int \frac{1}{e^x+1} = \int \frac{1}{\theta+1}$ $\quad C = 1,\ D = \theta + 1,\ D' = \theta$
$F(\theta) = \mathbb{Q}(x)(e^x)$

$$R(z) = \mathrm{res}(1 - z\theta, \theta + 1) = \det\left(\begin{bmatrix} -z & 1 \\ 1 & 1 \end{bmatrix}\right)$$
$$= -z - 1 \Rightarrow c_1 = -1.$$
$$v_1 = \gcd(1+\theta, 1+\theta) = \theta + 1$$
$$g_1 = -(-1)\cdot 1 \cdot x = x$$

$$\int \frac{1}{e^x+1} = c_1 \log v_1 + g_1 = -\log(e^x+1) + x.$$