12.7 Exponential Extension: Polynomial Part

Let $F = K(x)(\theta_1,\dots,\theta_n)$, $\theta = e^w$, $w\in F$, $w'\neq 0$, θ is NOT alegraic over F.
Let $\bar{p} = p_{-\ell}\theta^{-\ell}+\dots+p_0+\dots+p_m\theta^m$ where $p_i\in F$, i.e., $\bar{p}\in F[\theta,\theta^{-1}]$.
By Liouville's Theorem, if $\int \bar{p}$ is elementary then

$$\int \bar{p}(\theta) = \underset{\in F(\theta)}{v_0(\theta)} + \sum c_i \log \underset{\in F(\theta)}{v_i(\theta)} \qquad \text{WLOG } v_i\in F[\theta],\ \theta\nmid v_i$$

$$= \bar{V}_0(\theta) + \frac{a(\theta)}{b(\theta)} + \sum c_i \log v_i(\theta) \qquad \bar{V}_0\in F[\theta,\theta^{-1}],\ a,b\in F[\theta],\ \theta\nmid b$$

$$\gcd(a,b)=1,\ \gcd\left(v_i,\frac{dv_i}{d\theta}\right)=1$$

$$\Rightarrow \underset{\in F[\theta,\theta^{-1}]}{\bar{p}(\theta)} = \underset{\in F[\theta,\theta^{-1}] \text{ by Th 12.3}}{\bar{V}_0(\theta)'} + \left(\frac{a(\theta)}{b(\theta)}\right)' + \sum c_i \frac{v_i(\theta)'}{v_i(\theta)} \quad \gcd=1 \text{ by Th 12.8}$$

If $\deg_\theta b>0$ The terms in the PFD of $)$ cannot cancel $\Rightarrow b\in F$.
If $\deg_\theta v_i>0$ then v_i^{-1} cannot cancel $\Rightarrow v_i\in F$.

$$\Rightarrow \int \bar{p}(\theta) = \bar{V}(\theta) + \sum c_i \log \underset{\in F}{v_i} \quad \text{where } \bar{V}(\theta) = \sum_{i=-k}^{j} \underset{\in F}{q_i}\theta^i$$

Since $(\Sigma L)'\in F$ and for $\theta=e^w$, $\theta' = w'\theta\neq 0$, for $i\neq 0$

$$(q_i\theta^i)' = q_i'\theta^i + q_i\cdot i w'\theta\cdot\theta^{i-1} = \underset{\neq 0 \text{ by Th 12.3}}{(q_i'+iq_iw')}\theta^i$$

It follows that $j=m$ and $k=\ell$ i.e.

$$\int p_{-\ell}\theta^{-\ell}+\dots+p_0+\dots+p_m\theta^m = \underset{\in F}{q_{-\ell}}\theta^{-\ell}+\dots+\underset{\in F}{q_0}+\dots+\underset{\in F}{q_m}\theta^m + \sum c_i\log v_i$$

Example $\int xe^{-x}+x+\frac{1}{x}+e^{2x} = \int x\theta^{-1}+\left(x+\frac{1}{x}\right)+\theta^2$

$F(\theta)=\mathbb{Q}(x)(e^x)$

$$= \underset{\in \mathbb{Q}(x)}{q_{-1}}\theta^{-1} + \frac{1}{2}x^2 + \underset{\in \mathbb{Q}(x)}{q_2}\theta^2 + \log x$$