Rings & Fields 2.2

A set R with two binary operations $+$ and $\cdot$ is called a <u>ring</u> if $\forall a,b,c \in R$

(i) $a+b \in R$ (ii) $a \cdot b \in R$

(iii) $a+(b+c) = (a+b)+c$ (iv) $a\cdot(b\cdot c) = (a\cdot b)\cdot c$

(v) $\exists\, 0 \in R$ s.t. $0+a=a$ (vi) $\exists\, 1 \in R$ s.t. $1\cdot a = a \cdot 1 = a$, $1 \neq 0$.

(vii) $a+b = b+a$

(viii) $\exists\, -a \in R$ s.t. $a+(-a) = 0$ and

(ix) $a\cdot(b+c) = a\cdot b + a \cdot c$ and $(a+b)\cdot c = a\cdot c + b \cdot c$ hold.

Remark: Most texts do not require (vi).

Definition: If R is a ring and $a\cdot b = b \cdot a$ for all $a,b \in R$ then R is called a <u>commutative ring</u>.

Definition: If $a \in R$ and $\exists\, b \in R$ s.t. $a\cdot b = b \cdot a = 1$ Then a is called a <u>unit</u> or <u>invertible</u> in R and b is called The inverse of a denoted a^{-1}.

Definition: If R is a commutative ring and every non-zero element of R is invertible Then R is called a <u>field</u>.

Example: $\mathbb{Z}$ is a commutative ring.
The units in $\mathbb{Z}$ are $\{1, -1\}$
Hence $\mathbb{Z}$ is not a field.