Math 152 <u>Sigma Notation</u> $\sum_{i=1}^{n}(2i-1)$

Def: $$\sum_{i=m}^{n} f(i) = f(m)+f(m+1)+f(m+2)+\cdots+f(n)$$

Summand

E.g $$\sum_{i=1}^{n} 2i+3 = \underset{i=1}{5} + \underset{i=2}{7} + \underset{i=3}{9} + \cdots + \underset{i=n}{2n+3} = ?$$

(P1) $$\sum_{i=m}^{n}[f(i)+g(i)] = \sum_{i=m}^{n} f(i) + \sum_{i=m}^{n} g(i)$$

(P2) $$\sum_{i=m}^{n} c\cdot g(i) = c\sum_{i=m}^{n} g(i)$$

$$\sum_{i=m}^{m} c\,g(i) = c\,g(m) + c\cdot g(m+1) + \cdots + c\,g(n)$$
$$= c\,[\underbrace{g(m)+g(m+1)+\cdots}\; g(n)] = c\cdot\sum_{i=m}^{n} g(i).$$

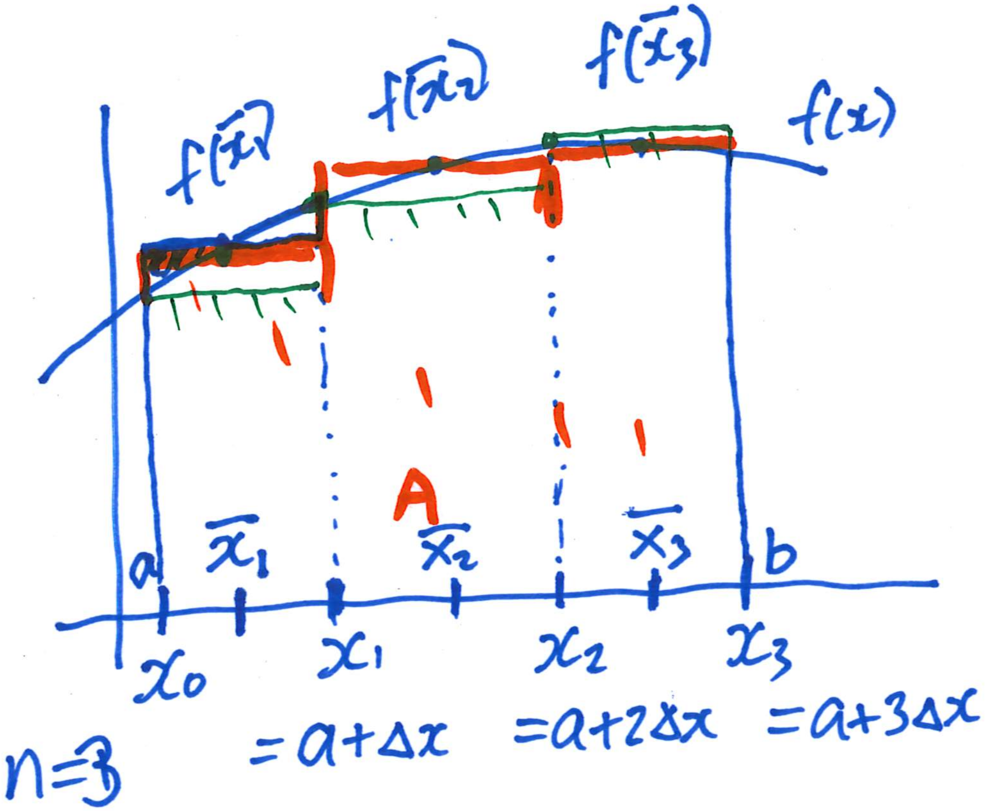
(P3) $$\sum_{i=1}^{n} 1 = 1+1+\cdots+1 = n\cdot 1$$

(P4) $$\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

(P5) $\displaystyle\sum_{i=1}^{n} i^2 = 1^2+2^2+\cdots+n^2 = \frac{1}{6}n(n+1)(2n+1).$

Eg. $\displaystyle\sum_{i=1}^{n}(2i+3) \overset{(P1)}{=} \sum_{i=1}^{n} 2i + \sum_{i=1}^{n} 3 \overset{(P2)}{=} 2\sum_{i=1}^{n} i + 3\sum_{i=1}^{n} 1 = 2\frac{n(n+1)}{2} + 3\cdot n.$

5.1 The Midpoint Rule M_n



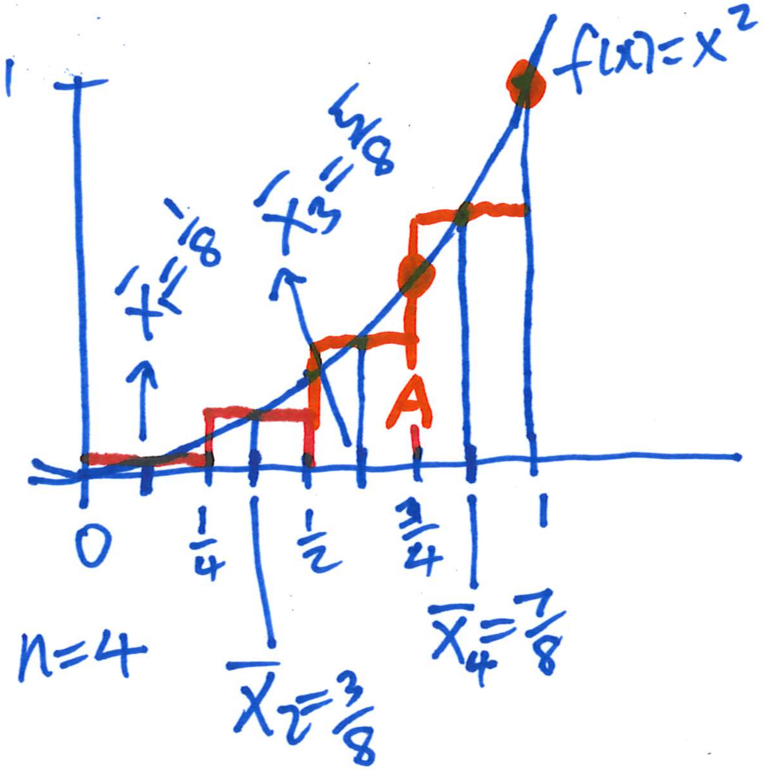
Divide $[a,b]$ into n subintervals $[x_0,x_1], [x_1,x_2], \ldots, [x_{n-1},x_n]$ of equal width $\Delta x = (b-a)/n$.

Let $\bar{x}_i = (x_{i-1}+x_i)/2$ for $i=1,2,\ldots,n$.

Approximate A with n rectangles

$$M_n = \Delta x f(\bar{x}_1) + \Delta x f(\bar{x}_2) + \cdots + \Delta x f(\bar{x}_n)$$
$$= \Delta x \sum_{i=1}^{n} f(\bar{x}_i).$$

Example



$n=4$

$\Delta x=\frac{1}{4}$

$$M_4=\frac{1}{4}\left[f\left(\tfrac{1}{8}\right)+f\left(\tfrac{3}{8}\right)+f\left(\tfrac{5}{8}\right)+f\left(\tfrac{7}{8}\right)\right]$$

$$=\frac{1}{4}\left[\frac{1}{64}+\frac{9}{64}+\frac{25}{64}+\frac{49}{64}\right]=\frac{84}{256}=.328125$$

$$=.33$$

$$L_4=\frac{14}{64}=\cancel{.4\ldots}\ .21875$$

$$R_4=\frac{30}{64}=.41875$$

M_n is generally a much better approx. of A than R_n or L_n.

$$A=\lim_{n\to\infty}M_n=\lim_{n\to\infty}R_n=\lim_{n\to\infty}L_n$$

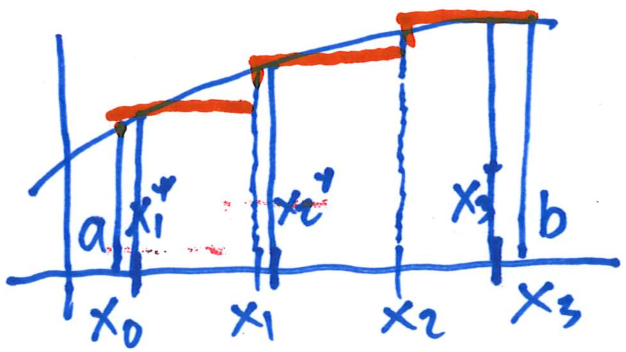
5.2 The definite integral.

Riemann. Pick x_i^* anywhere in $[x_{i-1}, x_i]$ i.e. $x_{i-1} \leq x_i^* \leq x_i$.

Approximate A with n rectangles

$$S_n = \sum_{i=1}^{n} \Delta x\, f(x_i^*) \quad \text{then} \quad \lim_{n\to\infty} S_n = A.$$

↑ Riemann sum.



$n=3$.

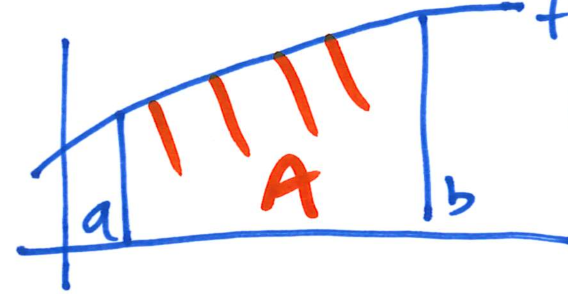
If $x_i^* = x_{i-1}$ then $S_n = L_n$

If $x_i^* = \overline{x_i}$ then $S_n = M_n$.

Leibniz. Let $f(x)$ be continuous on $[a,b]$. Define the definite integral

integration symbol

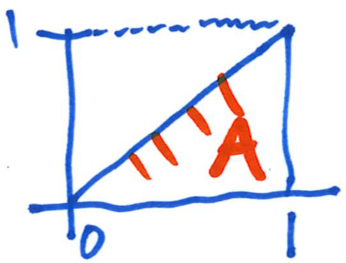
$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left[\sum_{i=1}^{n} \Delta x\, f(x_i^*)\right] =$$



$f(x) \geq 0$ on $[a,b]$

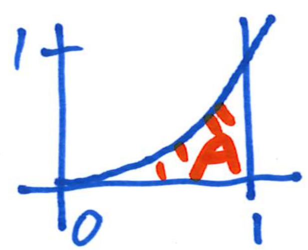
integrand

What is $\int_0^1 x\,dx =$



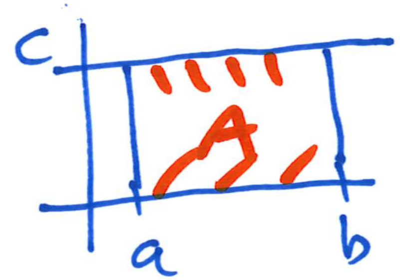
$= \frac{1}{2}$

What is $\int_0^1 x^2\,dx =$



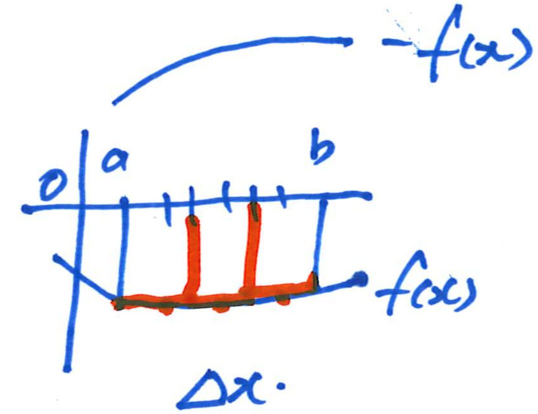
$= \frac{1}{3}$

What is $\int_a^b c\,dx =$



$= c(b-a)$

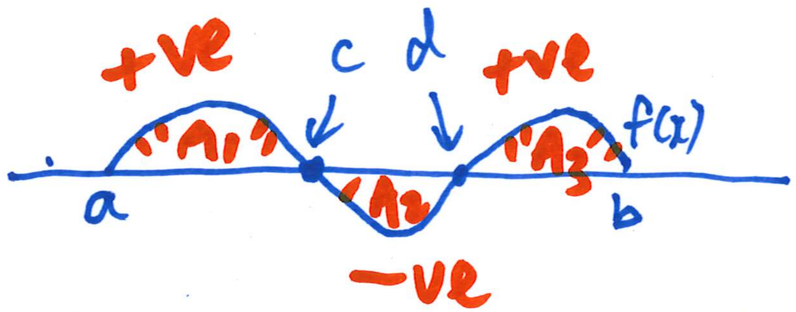
Properties of Definite Integrals



(1) If $f(x) > 0$ on $[a,b]$ then $\int_a^b f(x)\,dx > 0$

(2) If $f(x) < 0$ on $[a,b]$ then $\int_a^b f(x)\,dx < 0$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x\, \underbrace{f(x_i^*)}_{<0}$$

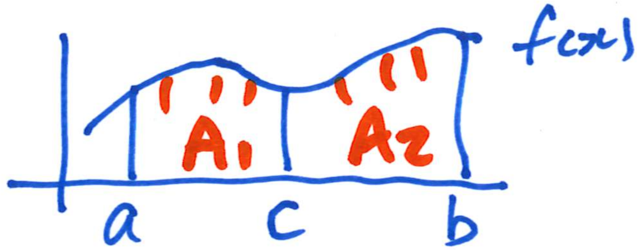


$$\int_a^b f(x)\,dx = A_1 - A_2 + A_3$$

(3) $\int_a^b \left(f(x)+g(x)\right)dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(4) $\int_a^b c\cdot f(x)\,dx = c\int_a^b f(x)\,dx$ $\qquad \int_a^b c\, f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} c\, f(x_i^*)$

(5)

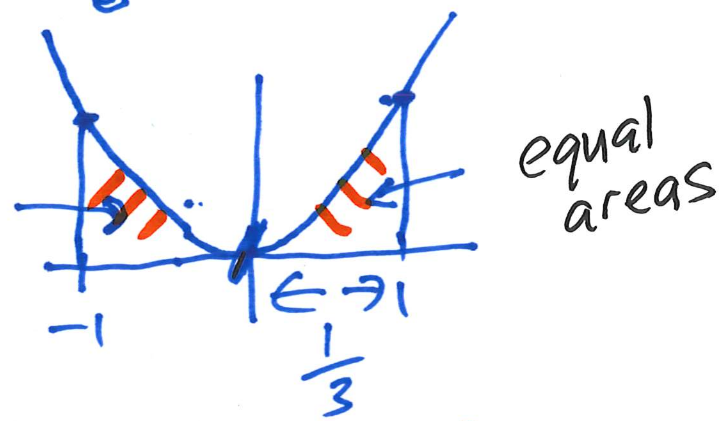


$$\int_a^b f(x)\,dx = \underbrace{\int_a^c f(x)\,dx}_{A_1} + \underbrace{\int_c^b f(x)\,dx}_{A_2}$$

Calculate $\int_0^1 (2x^2 - x)\,dx \overset{P3}{=} \int_0^1 2x^2\,dx + \int_0^1 -x\,dx \overset{P4}{=}$

$$= 2\int_0^1 x^2\,dx - \int_0^1 x\,dx$$

$$= 2\cdot\tfrac{1}{3} - \tfrac{1}{2} = \tfrac{1}{6}.$$



Calculate $\int_{-1}^1 x^2\,dx = \boxed{2}\int_0^1 x^2\,dx$

$$= 2/3.$$

Calculate $\int_0^{2\pi} \sin x\,dx = 0$