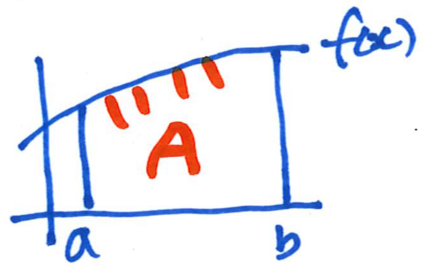
5.3 The Fundamental Theorem of Calculus

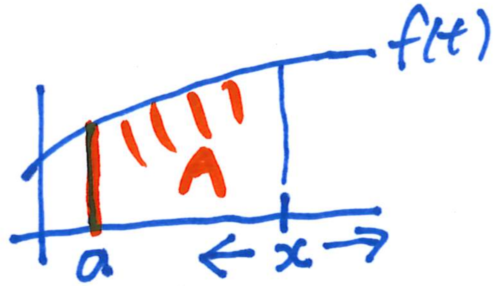
Video (34m) on 4.9 Antiderivatives on Canvas
Assignment #1 due next Tues/Wed.
For help go to the Calculus Workshop.

The Definite Integral



$$= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x\, f(x_i^*) = \int_a^b f(x)\,dx$$

Consider $g(x) = \int_a^x f(t)\,dt =$



$g(a) = 0$

E.g. $g(x) = \int_0^x 1\,dt$

$g(1) = 1$
$g(x) = x$
$g(0) = 0$

So $g(x)$ is an "area" function.

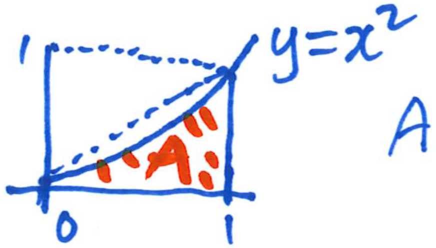
The F.T.C.

Let $f(x)$ be a continuous function on $[a,b]$.

Part (1) If $g(x) = \int_a^x f(t)\,dt$ then $g'(x) = f(x)$ i.e. $g(x)$ is an antiderivative of $f(x)$.

Part (2) $\int_a^b f(x)\,dx = F(b) - F(a)$ where $F'(x) = f(x)$ i.e. $F(x)$ is an anti der. of $f(x)$.

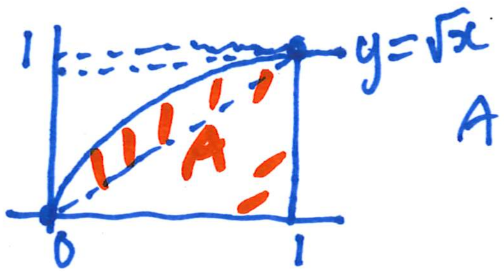
Example



$$A = \int_0^1 x^2\,dx = F(1) - F(0) = \tfrac{1}{3}\cdot 1^3 - \tfrac{1}{3}\cdot 0^3 = \tfrac{1}{3}.$$

$$F(x) = \tfrac{1}{3}x^3 \qquad F'(x) = \tfrac{1}{3}(3x^2) = x^2$$

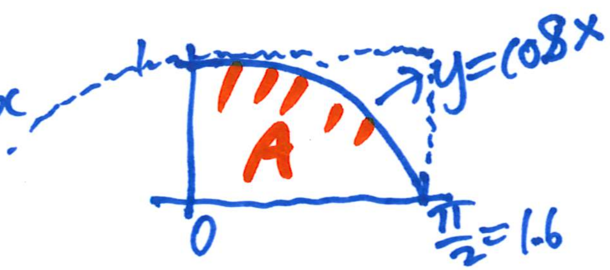
Ex 2.



$$A = \int_0^1 \sqrt{x}\,dx = F(1) - F(0) = \tfrac{2}{3}\cdot 1^{\frac{3}{2}} - \tfrac{2}{3}\cdot 0^{\frac{3}{2}} = \tfrac{2}{3} - 0 = \tfrac{2}{3}.$$

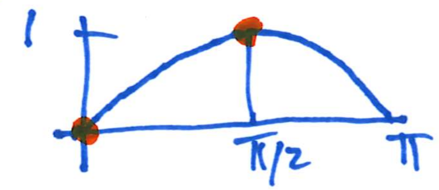
$$f(x) = \sqrt{x} = x^{\frac{1}{2}} \qquad F(x) = \tfrac{2}{3}x^{\frac{3}{2}} \qquad F'(x) = \tfrac{2}{3}\cdot\tfrac{3}{2}x^{\frac{1}{2}}$$

Ex

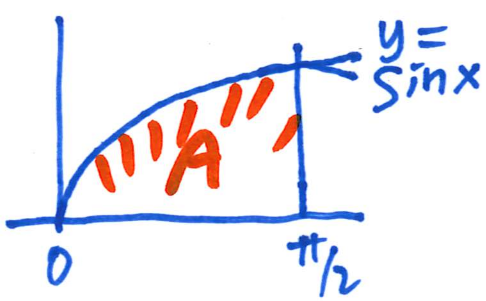


$$A = \int_0^{\frac{\pi}{2}} \cos x\,dx = F(\tfrac{\pi}{2}) - F(0) = \sin\tfrac{\pi}{2} - \sin 0 = 1 - 0 = 1.$$

$$F(x) = \sin x$$



Ex



$$A = \int_0^{\pi/2} \sin x\,dx = F(\tfrac{\pi}{2}) - F(0) = (-\cos\tfrac{\pi}{2}) - (-\cos 0)$$

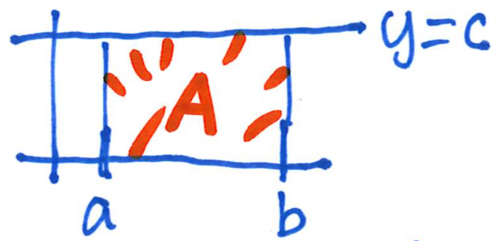
$$= -0 + 1 = 1.$$

$$F(x) = -\cos x$$

Notation. $\quad \left[F(x)\right]_a^b = F(b)-F(a) \qquad F(x)\Big]_a^b = F(b)-F(a)$

$$\int_0^1 x^2\,dx = \left[\tfrac{1}{3}x^3\right]_0^1 = \underbrace{\tfrac{1}{3}\cdot 1^3}_{x=1} - \underbrace{\tfrac{1}{3}\cdot 0^3}_{x=0} = \tfrac{1}{3}.$$

Example



$$A = \int_a^b c\,dx = \left[\,c\cdot x\,\right]_a^b = c\cdot b - c\cdot a$$

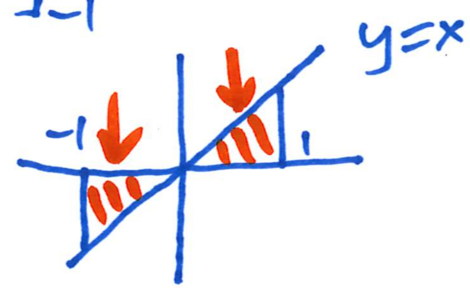
$$A = c\cdot(b-a)$$

Example

$$\int_0^1 (2x^2-x)\,dx = \left[\tfrac{2}{3}x^3 - \tfrac{1}{2}x^2\right]_0^1 = \left(\tfrac{2}{3}\cdot 1^3 - \tfrac{1}{2}\cdot 1^2\right) - \left(\tfrac{2}{3}\cdot 0^3 - \tfrac{1}{2}\cdot 0^2\right)$$
$$= \tfrac{2}{3} - \tfrac{1}{2} - 0 = \tfrac{1}{6}.$$

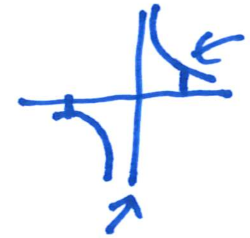
Example

$$\int_{-1}^1 x\,dx = \left[\tfrac{1}{2}x^2\right]_{-1}^1 = \tfrac{1}{2}\cdot 1^2 - \tfrac{1}{2}\cdot(-1)^2 = \tfrac{1}{2} - \tfrac{1}{2} = 0.$$



$$A = \int_0^1 x\,dx - \int_{-1}^0 x\,dx.$$

The F.T.C. Let $f(x)$ be continuous on $[a,b]$



$y=\frac{1}{x}$

$[-1,1]$

Part (1) If $\boxed{g(x)=\int_a^x f(t)\,dt}$ then $\boxed{g'(x)=f(x)}$ $\Downarrow$ $\Uparrow$

Part (2) If $\boxed{F'(x)=f(x)}$ then $\int_a^b f(x)\,dx = \underline{F(b)-F(a)}$.

Proof. (1) $\Rightarrow$ (2) $F(x)$ and $g(x)$ are antiderivatives of $f(x)$ $\Rightarrow$ $F(x)=g(x)+C$.

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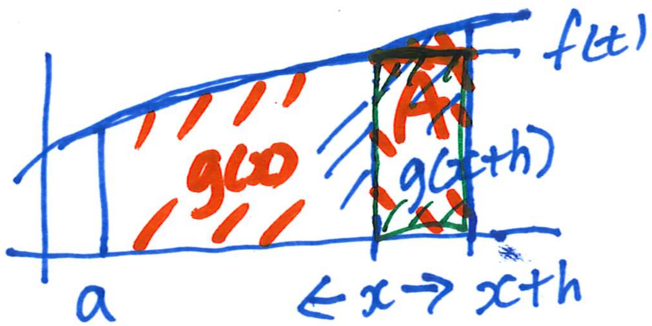
$x=b$ $x=a$

$$F(b)-F(a) = (g(b)+\cancel{C}) - (g(a)+\cancel{C})$$
$$= g(b)-g(a).$$
$$= \int_a^b f(t)\,dt - \underbrace{\int_a^a f(t)\,dt}_{=0} = \int_a^b f(t)\,dt = \int_a^b f(x)\,dx$$

Proof of Part (1) If $g(x)=\int_a^x f(t)\,dt$ then $g'(x)=f(x)$.

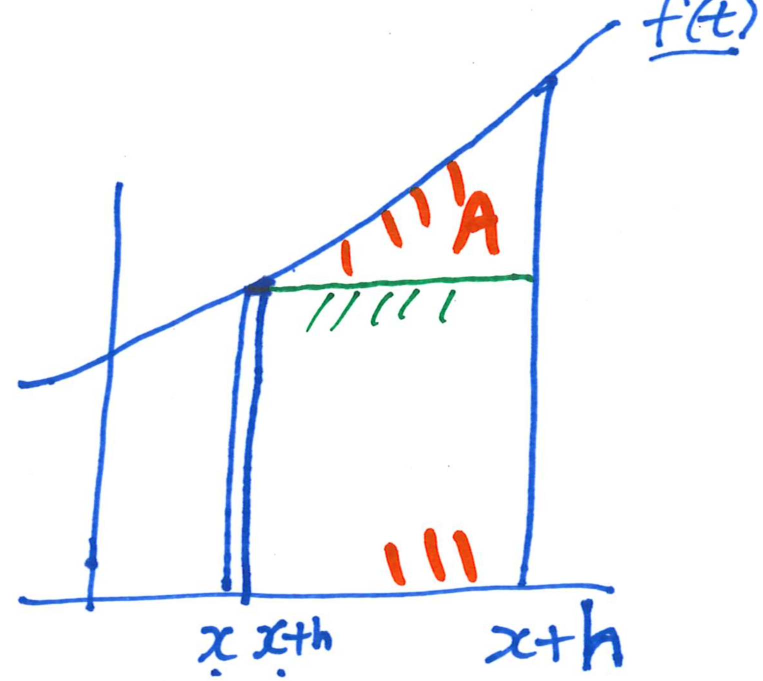


$A = g(x+h) - g(x) \approx h\,f(x)$ ← area of the rectangle

$\Rightarrow \dfrac{g(x+h)-g(x)}{h} \approx f(x)$.

$$\lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right] = f(x)$$

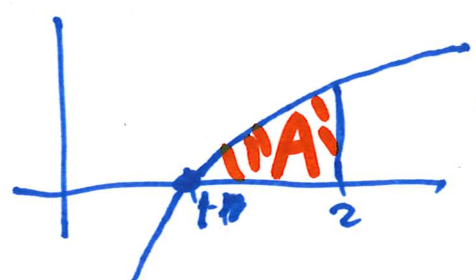
By def $g'(x)$



The area $A=\int_a^b f(x)\,dx = F(b)-F(a)$ only depends on two points of F.

We need to know an antiderivative of $f(x)$.

$\int_1^2 \ln x \, dx =$



What is an antiderivative of $\ln x$?