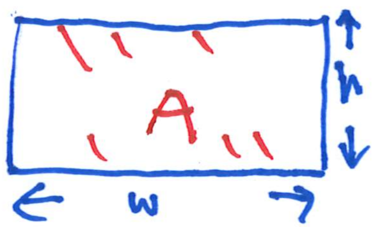
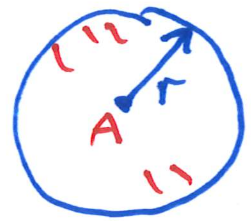
5.1 Areas and Distances

- The Calculus Workshop Opens this week. See Canvas for hours.

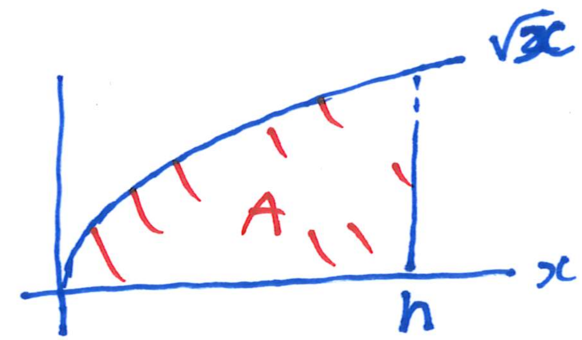


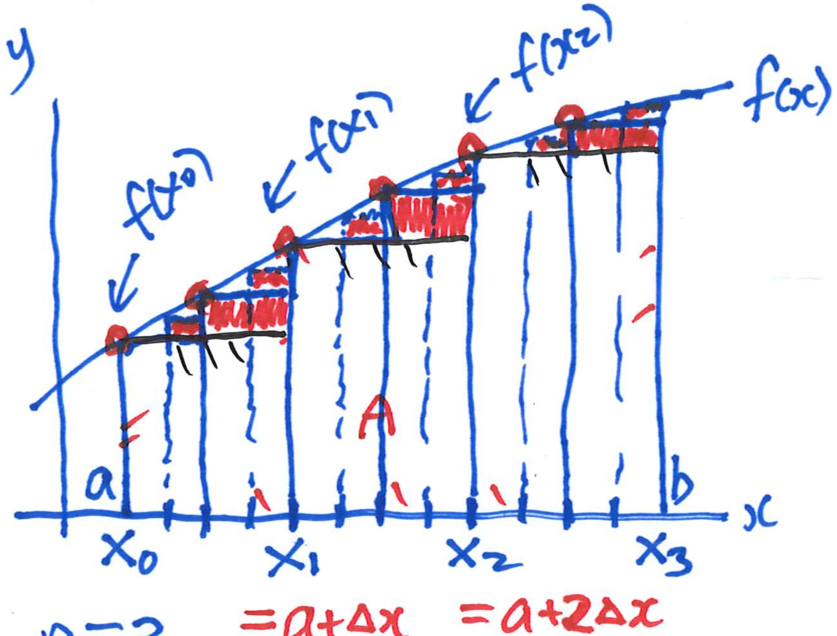
$A = w \cdot h$

F.T.C.



$A = \pi r^2$





$n=3$
$n=6$
$n=12$
$n=24$.

Let A be the area under $f(x)$, above the x axis between $x=a$ and $x=b$.

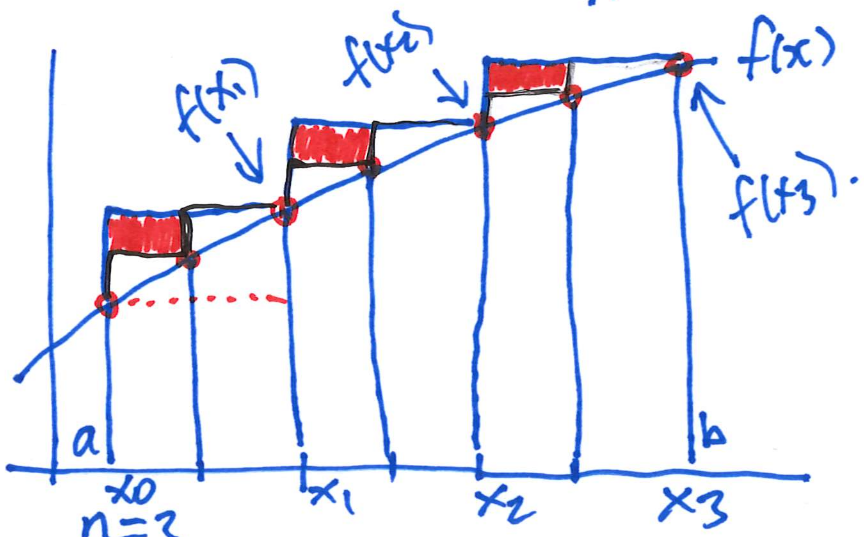
Divide $[a,b]$ into n subintervals $[x_0,x_1], [x_1,x_2], \ldots, [x_{n-1},x_n]$ of equal width $\Delta x = (b-a)/n$ so that $x_i = a + i\Delta x$

Let L_n be the area of the n rectangles.

$$L_n = \Delta x f(x_0) + \Delta x f(x_1) + \cdots + \Delta x f(x_{n-1}) = \sum_{i=0}^{n-1} \Delta x f(x_i) = \Delta x \sum_{i=0}^{n-1} f(x_i).$$

$$A = \lim_{n\to\infty} L_n$$

Let R_n be the area of these n right rectangles.

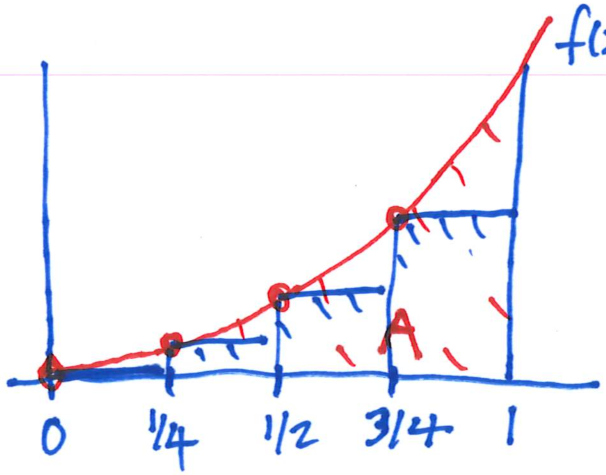


$n=6$

$$R_n = \Delta x f(x_1) + \Delta x f(x_2) + \cdots + \Delta x f(x_n)$$
$$= \sum_{i=1}^{n} \Delta x f(x_i) = \Delta x \sum_{i=1}^{n} f(x_i).$$

$$A = \lim_{n\to\infty} R_n.$$

Example



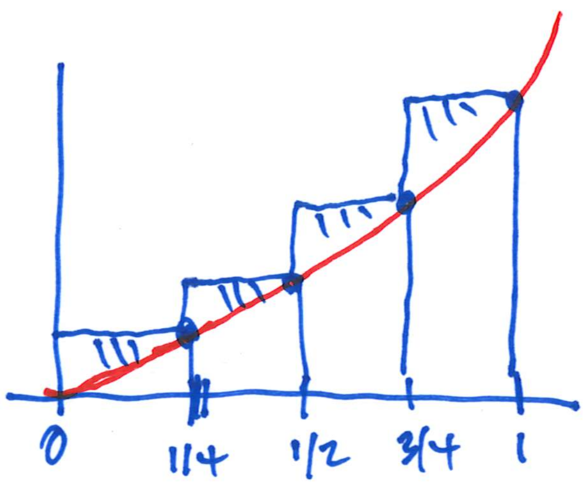
$n=4 \quad \Delta x = 1/4$

$$L_4 = \tfrac{1}{4}\left(f(0)+f(\tfrac{1}{4})+f(\tfrac{1}{2})+f(3/4)\right)$$

$$= \tfrac{1}{4}\left(0^2+(\tfrac{1}{4})^2+(\tfrac{1}{2})^2+(3/4)^2\right)$$

$$= \tfrac{1}{4}\left(\frac{0+1+4+9}{16}\right) = \frac{14}{64} = 0.21875.$$

$$L_{1000} = 0.33283$$



$$R_4 = \tfrac{1}{4}\left(f(\tfrac{1}{4})+f(\tfrac{1}{2})+f(3/4)+f(1)\right)$$

$$= 1/4\left(\frac{1+4+9+16}{16}\right) = \frac{30}{64} = 0.46875$$

$$R_{1000} = 0.33383$$

$$L_{1000} = 0.33283 < \underset{\overset{\shortparallel}{1/3\,?}}{A} < 0.33383$$

Recall $\displaystyle\sum_{i=1}^{n} i^2 = 1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6} = \boxed{\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}}$

$f(x)=x^2$ on $[\overset{a}{0},\overset{b}{1}]$

$\Delta x = 1/n \qquad x_i = 0 + i/n.$

$$R_n = \frac{1}{n}\Big(\underset{x_1}{f(\tfrac{1}{n})}+\underset{x_2}{f(\tfrac{2}{n})}+\cdots+\underset{x_n}{f(\tfrac{n}{n})}\Big) = \frac{1}{n}\Big(\big(\tfrac{1}{n}\big)^2+\big(\tfrac{2}{n}\big)^2+\cdots+\big(\tfrac{n}{n}\big)^2\Big)$$

$$= \frac{1}{n}\left(\frac{\boxed{1^2+2^2+\cdots+n^2}}{n^2}\right)$$

$$= \frac{1}{n^3}\left(\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right)$$

$$= \boxed{\frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}}$$

$$A = \lim_{n\to\infty} R_n$$

$$= \lim_{n\to\infty}\Big(\frac{1}{3}+\underset{\downarrow 0}{\frac{1}{2n}}+\underset{\downarrow 0}{\frac{1}{6n^2}}\Big)$$

$$= 1/3$$

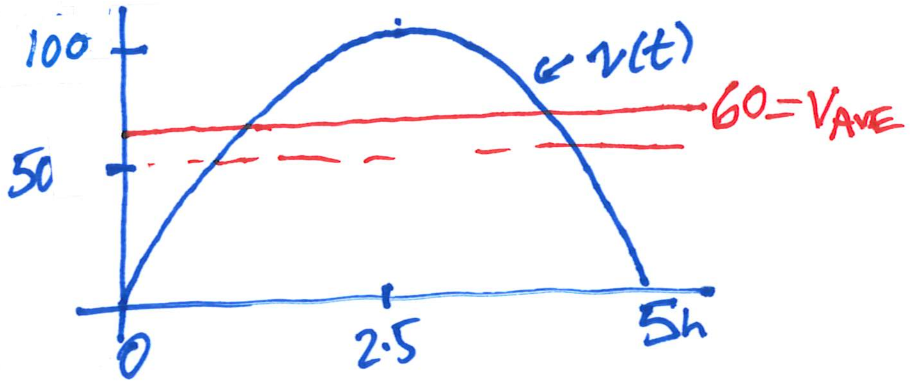
$$L_n = \frac{1}{n}\Big(\overset{x_0}{f(\tfrac{0}{n})}+\overset{x_1}{f(\tfrac{1}{n})}+\overset{x_2}{f(\tfrac{2}{n})}+\cdots+\overset{x_{n-1}}{f(\tfrac{n-1}{n})}\Big)$$

$$= \frac{1}{n}\left(\frac{0^2+[1^2+2^2+\cdots+(n-1)^2+n^2]-n^2}{n^2}\right) = \frac{1}{n^3}\left[\left[0+\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}\right]-n^2\right]$$

$$= \frac{1}{3}-\frac{1}{2n}+\frac{1}{6n^2}$$

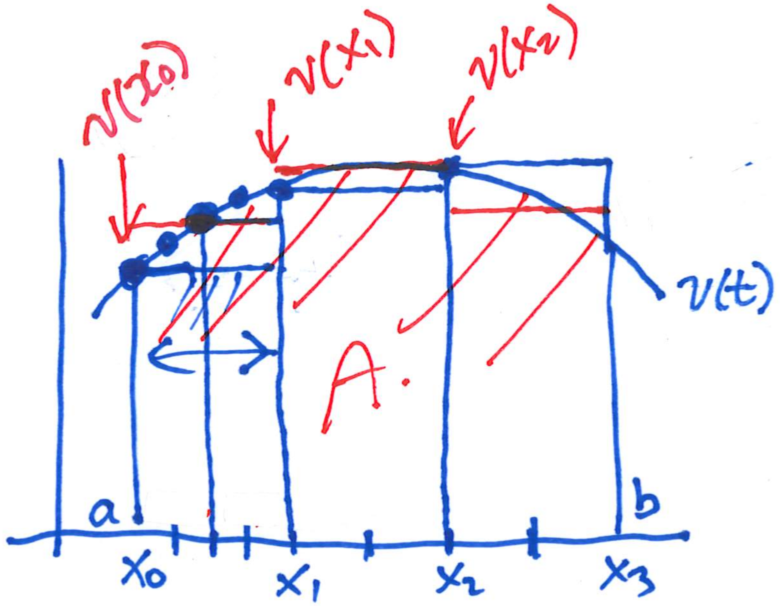
$$A = \lim_{n\to\infty}\left(\frac{1}{3}-\frac{1}{2n}+\frac{1}{6n^2}\right) = \frac{1}{3}.$$

The Distance Problem



Consider a car travelling at $v(t)$ kmph.
Let D be the distance travelled on $[a,b]$.
Let V_{AVE} be the average velocity on $[a,b]$

$$D = V_{AVE} \times \text{TIME}.$$
$$D = 60 \cdot 5 = 300\text{ km}.$$



$n=3$
$n=6$
$n=12$

Divide $[a,b]$ into n subintervals $[x_0, x_1], \ldots$

of equal width $\Delta x = (b-a)/n$

$$L_3 = v(x_0) \cdot \Delta x + v(x_1)\Delta x + v(x_2)\Delta x \approx D$$

estimate for the average velocity on $[x_0, x_1]$.

Time

$$\lim_{n\to\infty} L_n = D. \qquad \text{plausible??}$$

But $\displaystyle\lim_{n\to\infty} L_n = A.$ So $\underline{\underline{D = A.}}$