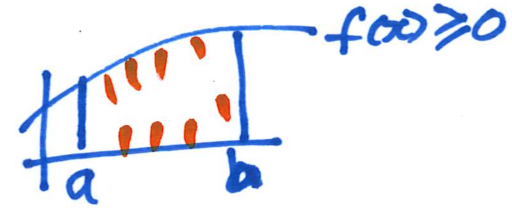
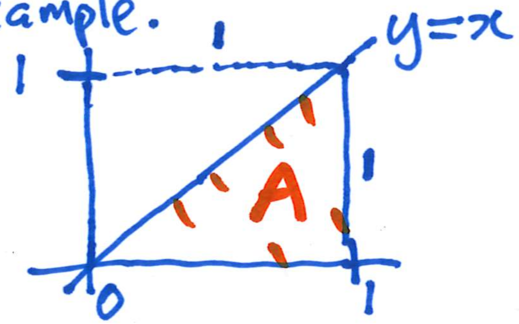
5.4 Indefinite Integrals and the Net Change Theorem.

Assignment #1 due Monday @ 11pm. Assignment #2 posted.

The FTC (2). Let $f(x)$ be continuous on $[a,b]$.

If $F'(x) = f(x)$ then $\int_a^b f(x)\,dx = F(b) - F(a) =$ 

Example.



$$A = \int_0^1 x\,dx = F(1) - F(0) = \left(\tfrac{1}{2}\cdot 1^2 + \cancel{3}\right) - \left(0 + \cancel{3}\right) = \tfrac{1}{2}.$$

$$F(x) = \tfrac{1}{2}x^2 + 3$$

Notation $\left[F(x)\right]_a^b = F(b) - F(a)$

$$A = \int_0^1 \underset{\substack{\uparrow \\ f(x)}}{x}\,dx = \Big[\underset{\substack{\uparrow \\ F(x)}}{\tfrac{1}{2}x^2}\Big]_0^1 = \overset{F(1)-F(0)}{\tfrac{1}{2}\cdot 1^2 - 0} = \tfrac{1}{2}$$

The traditional notation for an antiderivative (indefinite integral) of $f(x)$ is

$$\int f(x)\,dx$$

Examples ① $\int (2x+3)\,dx = x^2+3x+C$ ← general antiderivative, ← integration constant.

② $\int \ln x\,dx = x\ln x - x + C$

$$\frac{d}{dx}(x\ln x - x + C) = 1\cdot\ln x + x\cdot\frac{1}{x} - 1 + 0 = \ln x \quad ✓$$

Notice $\frac{d}{dx}\int f(x)\,dx = f(x)$.

Is $\int (f(x)+g(x))\,dx = \underline{\int f(x)\,dx + \int g(x)\,dx}$? Yes.

$$\frac{d}{dx}\left(\int f(x)\,dx + \int g(x)\,dx\right) = \frac{d}{dx}\int f(x)\,dx + \frac{d}{dx}\int g(x)\,dx$$

$$\begin{array}{ccc} \parallel & & \parallel \\ f(x) & + & g(x) \end{array}$$

Table of indefinite integrals (general antiderivatives) on page 410

① $\int c\cdot f(x)\,dx = c\int f(x)\,dx$ ② $\int (f(x)\pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

③ $\int k\,dx = kx + C$ ④ $\int \frac{1}{x}\,dx = \ln x + C$

⑤ $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$, $n \neq -1$ e.g. $\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = \frac{1}{-2+1}x^{-1} = -\frac{1}{x} + C.$

$\int \sin x\,dx = -\cos x + C$ $\int \cos x\,dx = \sin x + C$ $\int e^x\,dx = e^x + C$

$\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C$ $\boxed{\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C}$ $\int \ln x\,dx = x\ln x - x + C$

Example. $\int (10x^4 + 2\cos x)\,dx \overset{②}{=} \int 10x^4\,dx + \int 2\cos x\,dx$

$= 10\int x^4\,dx + 2\int \cos x\,dx = 10\left(\frac{1}{5}x^5 + C_1\right) + 2(\sin x + C_2)$

$= 2x^5 + 2\sin x + \boxed{10C_1 + 2C_2} = C$

$= 2x^5 + 2\sin x + C$ ✓

Example. $\displaystyle\int \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt = \int \left(2 + \sqrt{t} - t^{-2}\right) dt$ $\sqrt{t} = t^{1/2}$

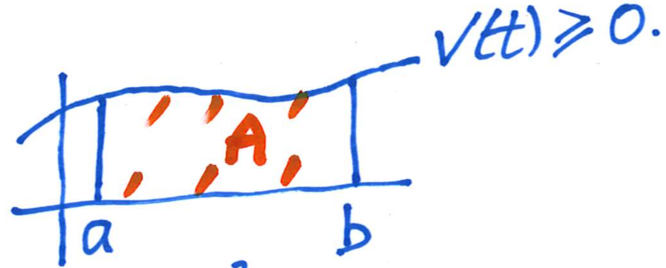
$$= 2t + \frac{2}{3}t^{3/2} - \left(-\frac{1}{t}\right) + C$$

$$\int 3x(4-x)\,dx = \int (12x - 3x^2)\,dx = 6x^2 - x^3 + C$$

The Distance Problem

FTC(2) $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$ where $\boxed{F'(x) = f(x)}$

$\displaystyle\int_a^b F'(x)\,dx = F(b) - F(a)$ Net Change Theorem.



Let $v(t)$ be the velocity of a car at time t.
Let $d(t)$ be the distance travelled at time t.

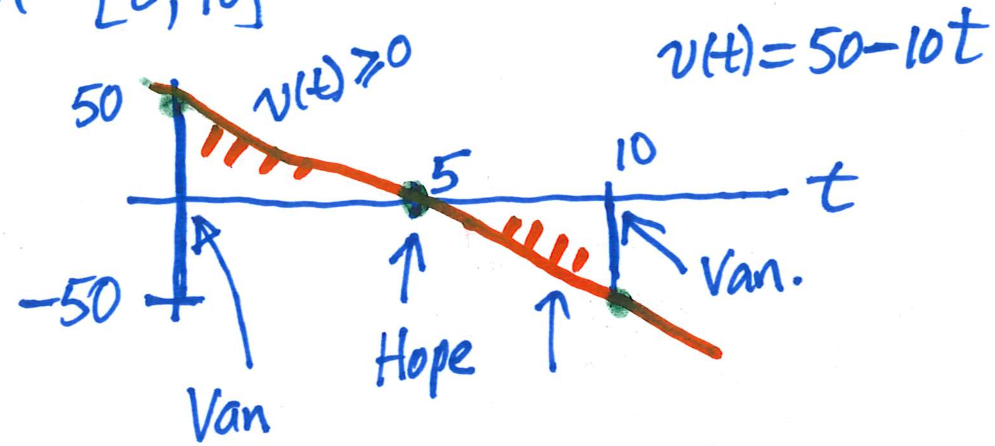
Then $\displaystyle A = \int_a^b v(t)\,dt = \int_a^b d'(t)\,dt = \left[d(t)\right]_a^b = d(b) - d(a) \stackrel{?}{=}$ distance travelled.

$v(t) = d'(t)$

Example Let $v(t) = 50 - 10t$ kmph on $[0, 10]$

$$\int_0^{10} (50 - 10t)\, dt = \left[50t - 5 \cdot t^2\right]_0^{10}$$

$$= (500 - 500) - 0 = 0 \text{ km.}$$



The integral $\int_a^b v(t)\, dt$ is the position (net change) at time $t = b$.

To calculate the distance D travelled use $D = \int_a^b |v(t)|\, dt$

$$D = \int_0^5 \underbrace{(50 - 10t)}_{v(t)}\, dt + \int_5^{10} \underbrace{(10t - 50)}_{-v(t)}\, dt = \left[50t - 5t^2\right]_0^5 + \left[5t^2 - 50t\right]_5^{10}$$

$$= (250 - 125) + \big(\underbrace{(500 - 500)}_{=0} - (125 - 250)\big)$$

$$= 125 + 125$$

$$= 250 \text{ km.}$$

Example If a company prints magazines at a rate (velocity) of $v(t) = 20/t$ thousands/hour how many are printed on $1 \leq t \leq 10$ hours (distance) (total production)

The production $P = \int_1^{10} \frac{20}{t}\,dt = 20\int_1^{10} \frac{1}{t}\,dt = 20\left[\ln t\right]_1^{10} = 20\ln 10 - 20\ln 1$

$= 20\ln 10 - 0$

$= 46.05$ thousand