The Euclidean Algorithm

Let E be a Euclidean domain with $v: E \setminus \{0\} \to \mathbb{N} \cup \{0\}$
Let $a, b \in E$, $b \neq 0$. Initialize $r_0 = a$ and $r_1 = b$.

$r_0 \div r_1$: $r_0 = q_2 r_1 + r_2$ $\quad r_2 \neq 0$ $\quad v(r_2) < v(r_1)$

$r_1 \div r_2$: $r_1 = q_3 r_2 + r_3$ $\quad r_3 \neq 0$ $\quad v(r_3) < v(r_2)$

$\vdots$

$r_{n-2} \div r_{n-1}$: $r_{n-2} = q_n r_{n-1} + r_n$ $\quad r_n \neq 0$ $\quad v(r_n) < v(r_{n-1})$

$r_{n-1} \div r_n$: $r_{n-1} = q_{n+1} r_n + r_{n+1}$ $\quad r_{n+1} = 0$

Claim r_n is a $\gcd(a,b)$.
Proof (i) Show $r_n | r_1 = b$ and $r_n | r_0 = a$
(ii) Show $d | r_0$ and $d | r_1 \Rightarrow d | r_n$
Claim n is finite (the algorithm terminates).
Proof $v(b) = v(r_1) > v(r_2) > v(r_3) > \cdots \geq 0$

Therefore a $\gcd(a, b \neq 0)$ exists in E.