Density of rational points on a family of del Pezzo surfaces of degree 1

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Quarantined Number Theory and Algebraic Geometry Seminar

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Rational points on varieties

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$X(k)$ set of *k-rational points* of X.

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Example

X a curve of genus at least 2, then $X(k)$ is finite.

Cubic surfaces

Example of del Pezzo surfaces: smooth cubic surfaces in $\mathbb{P}^3$.

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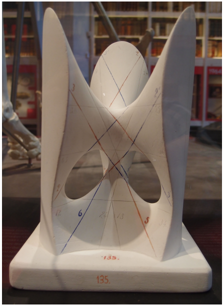
Example

$x^3 + y^3 + z^3 + w^3 = (x + y + z + w)^3$ (Clebsch surface)



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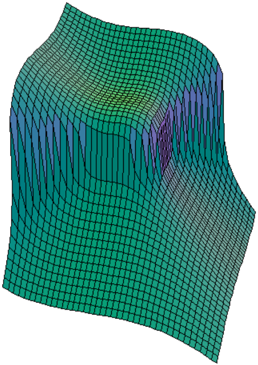
Example

$x^3 + y^3 + z^3 + w^3 = 0$ (Fermat cubic)



The geometry of cubic surfaces

Theorem (Cayley-Salmon, 1849)

- *A smooth cubic surface over an algebraically closed field contains exactly 27 lines.*
- *Any point on the surface is contained in at most three of those lines; such a point is an Eckardt point.*

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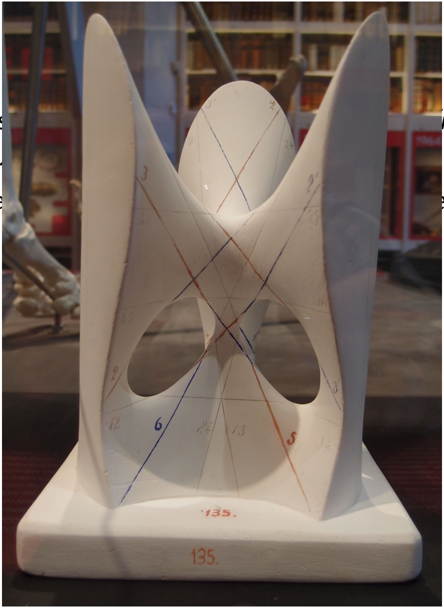

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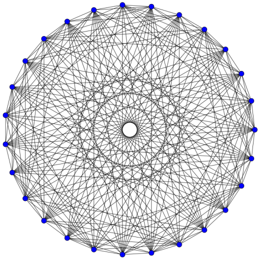

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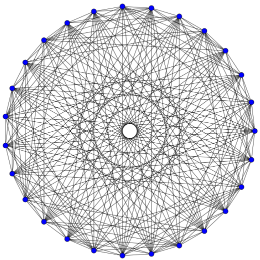

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Lemma (Hirschfeld, 1967)

There are at most 45 Eckardt points on a cubic surface.

Rational points on cubic surfaces

Let X be a smooth cubic surface over a field k.

Theorem (Segre, Manin, Kollár)

The following are equivalent.

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Property (ii) means that X is *unirational* over k. If k is infinite, this implies that $X(k)$ is dense in X.

Conclusion: k infinite, then

$$X(k) \neq \emptyset \text{ if and only if } X(k) \text{ dense in } X.$$

More general: del Pezzo surfaces

Definition

A *del Pezzo surface* X is a 'nice' surface with ample anticanonical divisor $-K_X$, i.e., X has an embedding in some $\mathbb{P}^n$, such that $-aK_X$ is linearly equivalent to a hyperplane section for some a. Its *degree* is the self intersection $(-K_X)^2$ of the anticanonical divisor.

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The degree is an integer between 1 and 9.

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Fact

The degree is an integer between 1 and 9.

Example

- Smooth cubic surfaces in $\mathbb{P}^3$ (degree 3).
- Complete intersection of two quadrics in $\mathbb{P}^4$ (degree 4).
- Double cover of $\mathbb{P}^2$, ramified over a smooth quartic curve (degree 2).
- For $3 \leq d \leq 9$, a del Pezzo surface is isomorphic to a surface of degree d in $\mathbb{P}^d$.

Geometry of del Pezzo surfaces

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Theorem

Let X be a del Pezzo surface of degree d over an algebraically closed field. Then X is isomorphic to either the product of two lines (then $d = 8$), or $\mathbb{P}^2$ blown up in $9 - d$ points in general position, where general position means

- *no three points on a line;*
- *no six points on a conic;*
- *no eight points on a cubic that is singular at one of them.*

Lines on a del Pezzo surface

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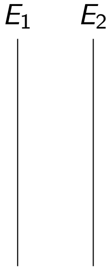
- the exceptional curves above $P_1, \ldots, P_r$;

  the strict transform of
- lines through two of the points;
- conics through five of the points;
- cubics through seven of the points, singular at one of them;
- quartics through eight of the points, singular at three of them;
- quintics through eight of the points, singular at six of them;
- sextics through eight of the points, singular at all of them, containing one of them as a triple point.

Degree 7

Blow up 2 points



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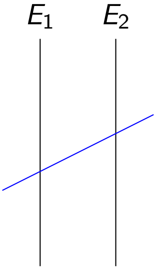

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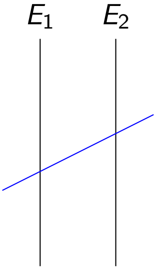
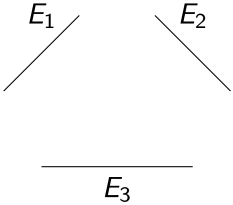


Degree 6

Blow up 3 points



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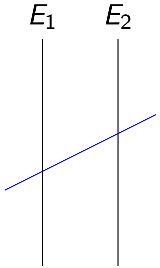

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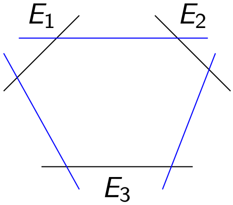

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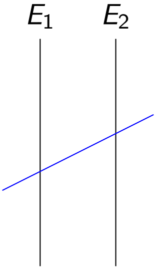

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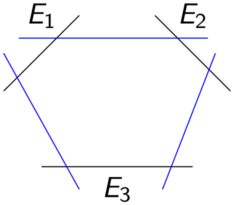
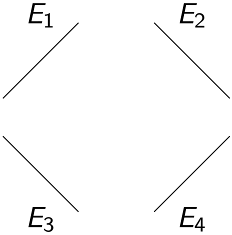


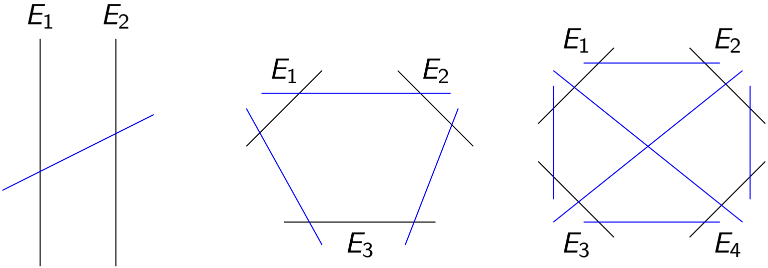
Degree 5

Blow up 4 points



Degree 7 | Degree 6 | Degree 5

Blow up 2 points | Blow up 3 points | Blow up 4 points



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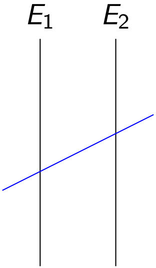

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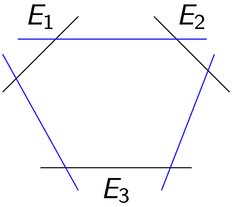

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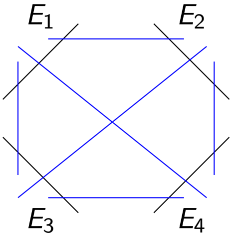

d	1	2	3	4	5	6	7	8
lines on X	240	56	27	16	10	6	3	1

Configuration of the lines on a del Pezzo surface

The intersection graph of the lines on a del Pezzo surface is known:

Degree 5: 10 lines, Petersen graph.
Degree 4: 16 lines, Clebsch graph.
Degree 3: 27 lines, complement of the Schläfli graph.

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Degree 2: 56 'lines', any line l intersects exactly one other line l' with multiplicity two, and 27 other lines with multiplicity one. These 27 lines do not intersect l', and they form again the complement of the Schläfli graph.
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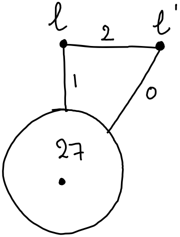

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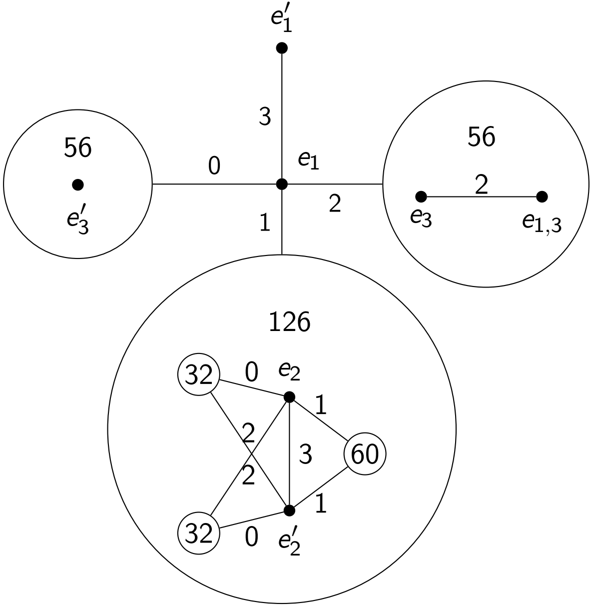
Degree 1: 240 'lines'...

Theorem (van Luijk-W.)

On a del Pezzo surface of degree 1, a point is contained in at most 10 lines in char $\neq 2, 3$, at most 16 lines in char 2, and 12 in char 3.

The intersection graph on the 240 lines



Rational points on del Pezzo surfaces

Recall:

• A variety X is *unirational* over a field k if there is a map $\mathbb{P}^n_k \dashrightarrow X$ for some n such that the image is dense in X.

• A smooth cubic surface has a k-rational point if and only if it is unirational over k.

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Theorem (Segre, Manin, Kollár, Pieropan)

A del Pezzo surface of degree $d \geq 3$ over a field k that has a k-rational point is unirational over k.

Rational points on del Pezzo surfaces

Del Pezzo surface of degree 2: double cover of $\mathbb{P}^2$, ramified over a smooth quartic curve.

Theorem (Salgado–Testa–Várilly-Alvarado)

A del Pezzo surface of degree 2 over a field k, that contains a k-rational point outside the ramification locus, that is not contained in the intersection of 4 lines, is unirational over k.

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Corollary

For these surfaces, under these conditions, if k is infinite then the set of k-rational points is dense.

What about del Pezzo surfaces of degree 1?

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• If X is not k-minimal (i.e. it contains a k-Galois orbit of pairwise disjoint exceptional curves), blow down X to obtain a del Pezzo surface X' of higher degree.
• Use previous theorems do determine if $X'(k)$ is dense in X'.
• Since X and X' are birationally equivalent, density of $X(k)$ follows from density of $X'(k)$.

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X minimal $\Rightarrow$ X has Picard rank 1 or 2.

What about del Pezzo surfaces of degree 1?

Theorem (Kollár-Mella, 2017)

A del Pezzo surface of degree 1 over a field k with $\text{char}\, k \neq 2$ that admits a conic bundle structure is unirational.

The surfaces in the theorem have Picard rank 2.

What about del Pezzo surfaces of degree 1?

Theorem (Kollár-Mella, 2017)

A del Pezzo surface of degree 1 over a field k with char $k \neq 2$ that admits a conic bundle structure is unirational.

The surfaces in the theorem have Picard rank 2.

Apart from this theorem, the question is wide open:

Q. Is there an example of a minimal del Pezzo surface of degree 1 with Picard rank 1 that is unirational?

Q. Is there an example of a minimal del Pezzo surface of degree 1 with Picard rank 1 that is **not** unirational?

A goal more within reach: density of rational points

Let X be a variety, k an infinite field.

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Q: X a del Pezzo surface of degree 1 over k. Is $X(k)$ dense in X with respect to the Zariski topology?

• We expect the answer to be yes.

• Partial results (Várilly-Alvarado '11, Ulas, Togbé, Jabara '07-'12, van Luijk–Salgado '14, Jardins-W.).

Summary: rational points on del Pezzo surfaces

X a del Pezzo surface of degree d over an infinite field k. Assume there is a point $P \in X(k)$.

d	k-unirational ($\mathbb{P}^n \dashrightarrow X$ dominant)	$\Rightarrow$	Zariski density of $X(k) \subset X$
≥ 3	✓		✓
2	P outside a closed subset		P outside a closed subset
1	char $k \neq 2$, Picard rank 2		several families

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Rest of this talk: explain the strategy in proving these partial results, and show new result (joint with Julie Desjardins).

Del Pezzo surfaces of degree 1

A del Pezzo surface X of degree 1 over a field k is isomorphic to a smooth sextic in the weighted projective space $\mathbb{P}(2,3,1,1)$ with coordinates $(x : y : z : w)$:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6,$$

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Such a surface always contains a k-rational point:

$$\mathcal{O} = (1:1:0:0).$$

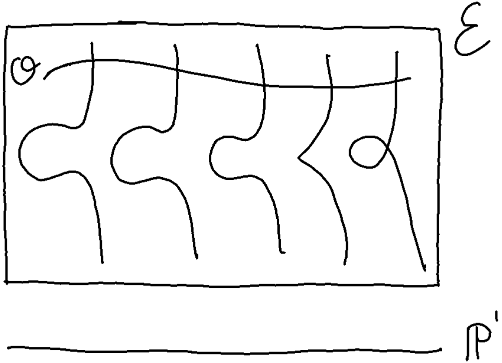
We have a map

$$X \dashrightarrow \mathbb{P}^1, \ \ (x:y:z:w) \longmapsto (z:w),$$

defined everywhere except in $\mathcal{O}$.

Del Pezzo surfaces of degree 1

When we blow up the point $\mathcal{O}$, we obtain an *elliptic surface*: a surface $\mathcal{E}$ with a morphism to $\mathbb{P}^1$, where almost all fibers are elliptic curves.



Example

Consider the del Pezzo surface given by

$$y^2 = x^3 + 27z^6 + 16w^6 \subset \mathbb{P}(2,3,1,1).$$

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Fiber above a point $(z_0 : w_0) \in \mathbb{P}^1$ is given by

$$y^2 = x^3 + 27z_0^6 + 16w_0^6,$$

isomorphic to an elliptic curve for allmost all $(z_0 : w_0)$.

Strategy to prove density of rational points

Recap:

X del Pezzo surface of degree 1 over a field k.

- X given by a smooth sextic in $\mathbb{P}(2,3,1,1)$.
- X contains a rational point $\mathcal{O}$.
- After blowing up $\mathcal{O}$, obtain an elliptic surface $\mathcal{E}$ with a dominant morphism $\mathcal{E} \longrightarrow X$.

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Idea: show that infinitely many fibers of $\mathcal{E}$ have infinitely many k-rational points - then $\mathcal{E}(k)$ lies dense in $\mathcal{E}$.

Showing that infinitely many fibers have infinitely many rational points - two techniques

1. Studying the *root number* of the fibers:

Theorem (Várilly-Alvarado,'11)

Let X be a del Pezzo surface given by

$$y^2 = x^3 + Az^6 + Bw^6,$$

with $A, B \in \mathbb{Z}$, such that either $3A/B$ is not a square, or $\gcd(A, B) = 1$ and $9 \nmid AB$. If the Tate-Shafarevich group of elliptic curves with j-invariant 0 is finite, then $X(\mathbb{Q})$ is dense in X.

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$\rightarrow$ Várilly-Alvarado showed that there are infinitely many disjoint pairs of fibers with opposite root number.

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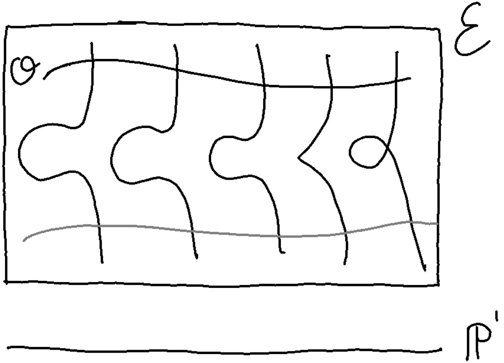
2. Creating a *multisection*:



Section

Showing that infinitely many fibers have infinitely many rational points - two techniques

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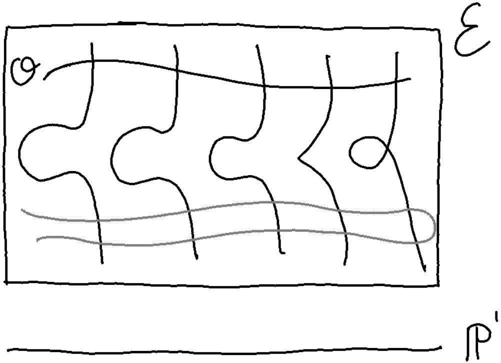

Multisection

Showing that infinitely many fibers have infinitely many rational points - two techniques

2. Creating a *multisection*:

Ulas, Togbé, Jabara '07-'12, van Luijk–Salgado '14.

Results for various families of del Pezzo surfaces of degree 1, by creating multisections of genus ≤ 1 and assuming that they contain infinitely many rational points.

Recent result

Let X be a del Pezzo surface of degree 1 over a number field k of the form

$$y^2 = x^3 + Az^6 + Bw^6, \tag{1}$$

with $A, B \in \mathbb{Z}$. Let $\mathcal{E}$ be the elliptic surface obtained by blowing up the point $(1 : 1 : 0 : 0)$.

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Theorem (Desjardins–W.)

The set $X(k)$ is dense in X if and only if the surface contains a point $P = (x_0 : y_0 : z_0 : w_0) \in X(k)$ such that $z_0, w_0 \neq 0$, and its corresponding point on $\mathcal{E}$ lies on a smooth fiber, and is non-torsion on this fiber.

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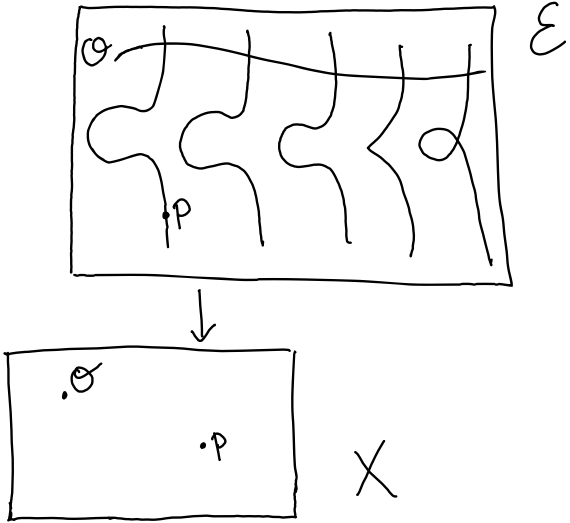

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This is the first result that gives necessary and sufficient conditions for the family (1) over any number field.

Idea of proof

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For $R = (x_R : y_R : z_R : w_R) \in X$, with $z_R, w_R \neq 0$, construct the curve on X cut out by

$$3x_R^2 z_R^2 xz - 2y_R z_R^3 y - (x_R^3 - 2Az_R^6)z^3 + 2Bz_R^3 w^3 = 0.$$

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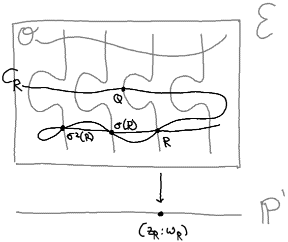
- Pull back to $\mathcal{E}$: curve C_R
- Automorphism $\sigma\colon (x : y : z : w) \longmapsto (x : y : z : \zeta_3^2 w)$
- C_R is a 3-section, and singular in R, $\sigma(R)$, $\sigma^2(R)$.



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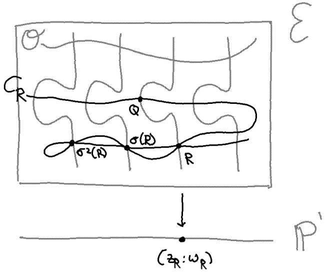
Goal: show that there is an R such that C_R intersects infinitely many fibers in a k-rational point that is non-torsion.



Proposition

If R is not contained in an exceptional curve on $\overline{X} = X \times_k \overline{k}$, then C_R either contains a section that is defined over k, or it is geometrically integral and has geometric genus at most 1, in which case R, $\sigma(R)$, $\sigma^2(R)$ are all double points.

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Recall: we assume there is $P = (x_0 : y_0 : z_0 : w_0) \in X(k)$ such that $z_0, w_0 \neq 0$, and its corresponding point on $\mathcal{E}$ lies on a smooth fiber $\mathcal{F}$, and is non-torsion on $\mathcal{F}$.

Corollary

There are infinitely many multiples R of P on $\mathcal{F}$ such that C_R either contains a section defined over k, or C_R has genus at most 1.

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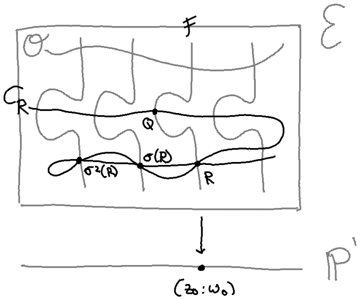
- *C_R contains a section over k: done.*
- *C_R geometric genus 0: it has infinitely many k-rational points.*
- *What about C_R with geometric genus 1?*

C_R geometrically integral of genus 1

V: set of multiples R on P on its fiber $\mathcal{F}$, such that C_R is geometrically integral of genus 1.

For $R \in V$, let Q be the third point of intersection of C_R with the fiber $\mathcal{F}$.

$E_R = (\tilde{C}_R, Q)$ elliptic curve, with point $D_R = \sigma(Q) + \sigma^2(Q)$.



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Proposition

For all but finitely many points in V, the point D_R has infinite order on E_R.

Finishing the argument

Conclusion: there is a point R on the fiber $\mathcal{F}$ such that C_R intersects infinitely many fibers of $\mathcal{E}$ in a k-rational point.

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Claim

Infinitely many of these points are non-torsion on their fiber.

• Upper bound $B = B(k)$ such that on all the fibers, all the torsion points have order at most B (Merel).

• For $m \leq B$ integer, let T_m be the zero locus of the m-th division polynomial $\psi_m \in k[x, y, t]$ of the generic fiber E of $\mathcal{E}$ over the function field $k(t)$.

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• For $m \leq B$ integer, let T_m be the zero locus of the m-th division polynomial $\psi_m \in k[x, y, t]$ of the generic fiber E of $\mathcal{E}$ over the function field $k(t)$.
• T_m intersects every smooth fiber of $\mathcal{E}$ in m^2 distinct points.
• C_R intersects the smooth fiber $\mathcal{F}$ in a point with multiplicity 2.
• So C_R is not contained in $\cup_{m \leq B} T_m$, hence intersects it in finitely many points.

Recap

We showed that infinitely many fibers of $\mathcal{E}$ contain a k-rational point of infinite order.

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Conversely: if $X(k)$ dense in X, then X contains a point P with infinite order on its fiber on $\mathcal{E}$; otherwise $X(k)$ would be contained in the torsion locus on $\mathcal{E}$, which is a closed subset (using Merel).

Example

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Explicitly:
- Two generators for $\mathcal{E}_{(1:1)}(\mathbb{Q})$ are given by $P_1 = (1:13:1:1)$ and $P_2 = (22:104:1:1)$ (magma).
- The curve C_{P_1} is cut out from X by $3xz - 26y + 323z^3 + 12w^3$.

Thank you!