CUBIC POINTS ON CUBIC CURVES AND THE BRAUER-MANIN OBSTRUCTION FOR K3 SURFACES

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Abstract. It is well-known that not all varieties over $\mathbb{Q}$ satisfy the Hasse principle. The famous Selmer curve given by $3x^3+4y^3+5z^3=0$ in $\mathbb{P}^2$, for instance, indeed has points over every completion of $\mathbb{Q}$, but no points over $\mathbb{Q}$ itself. Though it is trivial to find points over some cubic field, it is a priori not obvious whether there are points over a cubic field that is galois. We will see that such points do exist. K3 surfaces do not satisfy the Hasse principle either, which in some cases can be explained by the so-called Brauer-Manin obstruction. It is not known whether this obstruction is the only obstruction to the existence of rational points on K3 surfaces. We relate the two problems by sketching a proof of the following fact. If there exists a smooth curve over $\mathbb{Q}$ given by $ax^3+by^3+cz^3=0$ that is locally solvable everywhere, that has no points over any cubic galois extension of $\mathbb{Q}$, and whose Jacobian has trivial Mordell-Weil group, then the algebraic part of the Brauer-Manin obstruction is not the only one for K3 surfaces. No knowledge about K3 surfaces or Brauer-Manin obstructions will be assumed as known.

1. Obstructions to the existence of rational points

1.1. **Hasse principle.** The Hasse principle states for a variety X that if $X(\mathbb{Q}_p)$ is nonempty for all $p \le \infty$, then $X(\mathbb{Q})$ is nonempty. It is true for conics, not always true for cubics, e.g., $C\colon 3x^3+4y^3+5z^3=0$. The intersection of this curve with $L\colon 711x+172y+785z=0$ contains points over $\ell/\mathbb{Q}$ where ℓ is a Galois degree-3 extension of $\mathbb{Q}$.

1.2. **Brauer-Manin obstruction.** Let X be a smooth projective variety over a number field k. If $A \in \operatorname{Br} X := H^2_{\mathrm{et}}(X, \mathbb{G}_m)$, we get

$$\begin{tikzcd}
X(k) \arrow[r] \arrow[d, "f_A"'] & \prod_v X(k_v) = X(\mathbf{A}_k) \arrow[d] \arrow[dr, "\phi"] & & \\
\operatorname{Br} k \arrow[r] & \bigoplus_v \operatorname{Br} k_v \arrow[r] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Define $X(\mathbf{A}_k)^A = \phi^{-1}(0)$, so $X(k) \subseteq X(\mathbf{A}_k)^A$. If $X(\mathbf{A}_k) \neq \emptyset$ but $\bigcap_{A \in \operatorname{Br} X} X(\mathbf{A}_k)^A = \emptyset$, then one says that there exists a Brauer-Manin obstruction to the Hasse principle for X.

Theorem 1.1. *Suppose there is a cubic $C\colon ax^3+by^3+cz^3=0$ in $\mathbb{P}^2_{\mathbb{Q}}$ with $a,b,c \neq 0$ such that*

(1) *C has points over every completion of $\mathbb{Q}$.*
(2) *There does not exist a degree-3 Galois extension $\ell/\mathbb{Q}$ with $C(\ell) \neq \emptyset$.*

Then there exists a K3 surface Y over $\mathbb{Q}$ with $\operatorname{rk}\operatorname{Pic}\overline{Y} = 20$ such that for $\operatorname{Br}_1 Y := \ker\left(\operatorname{Br} Y \to \operatorname{Br}\overline{Y}\right)$ we have $\bigcap_{A\in\operatorname{Br}_1 Y} Y(\mathbf{A}_{\mathbb{Q}})^A \neq \emptyset$ and $Y(\mathbb{Q}) = \emptyset$.

Date: February 8, 2007.

We have

$$C \times C \simeq \{(P,Q,R) \in C \times C \times C : P+Q+R \text{ collinear}\}$$

where collinear means that $P+Q+R$ is linearly equivalent to a line section. The order-3 automorphism $\rho\colon (P,Q,R) \mapsto (Q,R,P)$ of the right hand side corresponds to an automorphism $\rho\colon (P,Q) \mapsto (Q,R)$ of $C \times C$ where R is the unique point of C such that $P+Q+R$ is linearly equivalent to a line section.

Let $X = C \times C/\rho$ and let Y be the minimal nonsingular model of X.

Theorem 1.2. *Let k be a number field. Let C be $ax^3+by^3+cz^3=0$ in $\mathbb{P}^2_k$, and define X and Y as above.*

(a) *If $C(\mathbf{A}_k) \neq \emptyset$, then $Y(\mathbf{A}_k) \neq \emptyset$.*

(b) *If there is no Galois degree-3 extension ℓ of k with $C(\ell) \neq \emptyset$, then $Y(k) = \emptyset$.*

(c) *If $C(\mathbf{A}_k) \neq \emptyset$ and $3 \nmid h(k)$ and $\frac{\operatorname{Br}_1 Y}{\operatorname{Br} k} \neq 0$, then there exists $u \in \mathcal{O}_k^\times$ such that $C \simeq x^3+uy^3+u^2z^3=0$ over k.*

Corollary 1.3. *Suppose $C(\mathbf{A}_k) \neq \emptyset$ and there does not exists a Galois degree-3 extension ℓ/k with $C(\ell) \neq \emptyset$ and $3 \nmid h(k)$ and for all u,*

$$C_u\colon x^3+uy^3+u^2z^3=0$$

does not satisfy $C_u(\mathbf{A}_k) \neq \emptyset$ or the condition that there does not exist a Galois degree-3 extension ℓ/k with $C_u(\ell) \neq \emptyset$. Then we have $\cap_{A \in \operatorname{Br}_1 Y} Y(\mathbf{A}_k)^A \neq \emptyset$, but $Y(k) = \emptyset$.

(a) Suppose $P \in C(k_v)$, then $(P,P) \in C \times C$ maps to some $R \in X(k_v)$. There is a diagram

$$\begin{tikzcd} C \times C \arrow[rr, "6"] \arrow[dr, "3"'] & & \check{\mathbb{P}}^2 \\ & X \arrow[ur, "2"'] & \end{tikzcd}$$

where the horizontal map sends (P,Q) to the line through P and Q. The map $X \to \check{\mathbb{P}}^2$ is ramified over $\check{C}$.

The affine curve $ax^3+by^3+c=0$ has a flex at $(0,-\alpha)$ where $\alpha = \sqrt[3]{c/b}$, and the tangent there is $y+\alpha z=0$, which corresponds to $[0:1:\alpha] \in \check{\mathbb{P}}^2$. The dual curve $\check{C}$ has 9 cusps. The singular points of X locally look like $y^2+x^3=z^2$. Blowing up once gives two $\mathbb{P}^1$'s above each, and their intersection will be a rational point if the singular point of X is rational.

Suppose $Q \in Y$ maps to R on X_C. Then R corresponds to a Galois-invariant orbit (P,P',P'') under ρ, so P is defined over some ℓ/k that is Galois of degree-3.

Poonen: The converse to (b) holds if $3J(k) = J(k)$, where $J = \operatorname{Jac} C$.

(c) $\frac{\operatorname{Br}_1 Y}{\operatorname{Br} k} \simeq H^1(k, \operatorname{Pic} \overline{Y})$. The 18 $\mathbb{P}^1$'s in Y lying above the 9 singular points of X give elements of $\operatorname{Pic}\overline{Y}$. Let r,s,t be the coordinates of $\check{\mathbb{P}}^2$. The strict transform D of $\sigma^* L_{r=0}$ on Y, where σ is the map $X \to \check{\mathbb{P}}^2$, is reducible since its self-intersection is -4, while irreducible D would have $D^2 \geq -2$. Similarly the pullback of $\sigma^* L_{s=0}$ and $\sigma^* L_{t=0}$ are irreducible, as well as the strict transforms of 9 more quadrics going through 6 cusps. These components give further elements of $\operatorname{Pic}\overline{Y}$, and together all these generate it. It follows that $\operatorname{Pic}\overline{Y}$ is defined over $k(\zeta,\alpha,\beta)$ where $\beta = \sqrt[3]{a/c}$. Let $\gamma = \alpha^{-1}\beta^{-1}$. The Galois group of $k(\zeta,\alpha,\beta)$ over k is a subgroup of $(\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}$.

Fact (based on checking all subgroups with Magma): $H^1(H, \operatorname{Pic}\overline{Y}) \neq 0$ implies that $\alpha/\gamma \in k$, which is equivalent to $ac/b^2 \in k^3$. In this case, C is isomorphic to $x^3+by^3+b^2z^3=0$.

Local solvability implies that b has all valuations divisible by 3, and with $3 \nmid h(k)$, this implies that b is a unit times a cube.