Math 842 Lecture 11. Lattices

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Remember that Milne defines a lattice to be a subgroup $\Lambda \subset \mathbb{R}^n$ that is generated by a set of $\mathbb{R}$-linearly independent generators, i.e., $\Lambda = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_r$, with $\{e_1, \ldots, e_r\}$ an $\mathbb{R}$-linearly independent set. It follows that a lattice must be finitely generated (and free, because it is a subgroup of a free abelian group), of rank $r \leq n$. If $r = n$ then we say that Λ is a *full lattice*. In that case, we can generate Λ with an $\mathbb{R}$-basis for $\mathbb{R}^n$.

We showed that *discrete* subgroups of $\mathbb{R}^n$ to are those that have finite intersection with bounded sets. The following result was perhaps not entirely satisfactorily proved (at least I wasn't completely happy with the exposition), so we'll give the proof here.

11.1 Theorem: *A discrete subgroup $\Lambda \subset \mathbb{R}^n$ is a lattice.*

Proof: Since we can replace $\mathbb{R}^n$ with the $\mathbb{R}$-vector space generated by Λ, which will be isomorphic to $\mathbb{R}^m$ for some $m \leq n$, we can assume without loss of generality that Λ generates $\mathbb{R}^n$. That means we can choose an $\mathbb{R}$-basis $\{v_1, \ldots, v_n\}$for $\mathbb{R}$ inside Λ. So, we already see that Λ contains a lattice, namely $v_1\mathbb{Z} + \cdots + v_n\mathbb{Z}$. We'll prove that Λ is a lattice itself using induction on n.

For $n = 0$ there is nothing to prove.

For $n = 1$, we see that Λ, being discrete, intersects sets of the form $\{v \in \mathbb{R}^1 : |v| < B\}$ in a finite set. Since $v_1 \in \Lambda$, we see that there are non-zero elements in there too. This means we can choose $e \in \Lambda$ such that $|e|$ is minimal among the non-zero elements in Λ.

Since $e_1\mathbb{R}^1 = \mathbb{R}^1$, we see that any element $w \in \Lambda$ can be written as $w = ae_1$ for a unique a. But then

$$w - \lfloor a \rfloor v_1 = (a - \lfloor a \rfloor)e_1 \in \Lambda$$

and we see that since $0 \leq (a - \lfloor a \rfloor) < 1$, that $|(a - \lfloor a \rfloor)e_1| < |e_1|$, so $a - \lfloor a \rfloor = 0$. It follows that $a \in \mathbb{Z}$, so $w \in e_1\mathbb{Z}$. This shows that $\Lambda = e_1\mathbb{Z}$, so indeed is a lattice.

Now the induction step. We consider the vector space $V = \langle v_1, \ldots, v_{n-1} \rangle \simeq \mathbb{R}^{n-1}$. Then $\Lambda' \cap V$ is a discrete subgroup of a space isomorphic to $\mathbb{R}^{n-1}$, so Λ' is a lattice (and Λ generates V, so it is a full rank lattice in there): we can write $\Lambda' = e_1\mathbb{Z} + \cdots + e_{n-1}\mathbb{Z}$, with $e_1, \ldots, e_{n-1}$ an $\mathbb{R}$-basis for V.

Since $v_1, \ldots, v_{n-1}$ span the same space as $e_1, \ldots, e_{n-1}$, we see that $e_1, \ldots, e_{n-1}, v_n$ is a basis for $\mathbb{R}^n$. We consider the exact sequence corresponding to the quotient $\mathbb{R}^n/V \simeq \mathbb{R}$.

$$0 \to V \to \mathbb{R}^n \xrightarrow{\phi} \mathbb{R} \to 0$$

The quotient map is realized by writing $w = a_1e_1 + \cdots + a_{n-1}e_{n-1} + a_nv_n$, and setting $\phi(w) = a_n$. We write $\Lambda'' = \phi(\Lambda)$. We'll prove that Λ'' is discrete. We need that Λ' is full rank for that!

Let's take a set $W = \Lambda'' \cap \{w \in \mathbb{R} : |w| < B\}$. We want to show it is empty. Take an element $v \in \Lambda$ with $\phi(v) = w$. Then we can write $w = a_1e_1 + \cdots + a_{n-1}e_{n-1} + av_n$. In fact, since $w' = \lfloor a_1 \rfloor e_1 + \cdots + \lfloor a_{n-1} \rfloor e_{n-1} \in \Lambda'$, we see that we can also use $w - w'$. But as you can see, $w - w'$ comes from a bounded set, so there are only finitely many candidates. Hence W is finite. This shows that Λ'' is a lattice, so $\Lambda'' = w_0\mathbb{Z}$. We can take $e_n \in \phi^{-1}(w_0) \cap \Lambda$. Then it's easy to check that $e_1, \ldots, e_n$ is indeed an $\mathbb{R}$-linearly independent set of generators of Λ. □