MATH843: Assignment 3

Due: Feb. 20, 2013

Name: ....

1. (Apostol Ch. 6, 17, 18): An arithmetical function f is called *periodic* with *period* k if $f(n) = f(m)$ for all n, m with $n \equiv m \pmod{k}$.

   (a) If f is periodic with period k, show that there is a smallest period $k_0 > 0$ for f and that $k_0 \mid k$.

   (b) Let f be periodic and completely multiplicative with smallest period k_0. Prove that $f(n) = 0$ whenever $\gcd(k, n) = 0$. Conclude that f must be a Dirichlet character modulo k.

   (c) Let χ be a Dirichlet character modulo k. Show that if k is squarefree then k is the smallest positive period for χ.

   (d) Give an example of a Dirichlet character modulo k for which k is not the smallest positive period

   (e) Give an example of a Dirichlet character for which the smallest positive period is not square-free.

2. (Montgomery-Vaughan 4.3 Q5) Let $\pi(x; q, a)$ denote the number of primes p with $p \leq x$ and $p \equiv a \pmod{q}$. Let

$$\theta(x; q, a) = \sum_{p \leq x \text{ and } p \equiv a \pmod{q}} \log p \text{ and } \psi(x; q, a) = \sum_{n \leq x \text{ and } n \equiv a \pmod{q}} \Lambda(n)$$

   The following is basically Chebyshev's result for primes in arithmetic progressions.

   (a) Show that

$$\theta(x; q, a) = \psi(x; q, a) + O(x^{1/2})$$

   (b) Show that

$$\pi(x; q, a) = \frac{\theta(x; q, a)}{\log x} + O\left(\frac{x}{(\log x)^2}\right)$$

   (c) Show that for any $C \geq 2$ and $x \geq C$ and $\gcd(a, q) = 1$, we have

$$\sum_{x/C < p \leq x; p \equiv a \pmod{q}} \frac{\log p}{p} = \frac{\log C}{\phi(q)} + O_q(1)$$

   (that's just a matter of looking at the final step in Dirichlet's proof and taking moderate care in the error terms)

   (d) Show that for any $q > 0$ and $\gcd(a, q) = 1$ there are constants C_q and $B_q > 0$ such that

$$\pi(x; q, a) > B_q \frac{x}{\log x} \text{ for } x \geq C_q$$

(e) Show that for $\gcd(a,q)=1$ we have

$$\liminf_{x\to\infty}\frac{\pi(x;q,a)}{x/\log x}\le 1/\phi(q)$$

and

$$\limsup_{x\to\infty}\frac{\pi(x;q,a)}{x/\log x}\ge 1/\phi(q)$$

3. (Montgomery-Vaughan, 4.3 Q8) Let $K=\mathbb{Q}(i)$ be the Gaussian field, let $\mathcal{O}_K=\mathbb{Z}[i]=\{a+ib: a,b\in\mathbb{Z}\}$ be the field of integers in K. Note that $\mathbb{Z}[i]$ is a principal ideal domain. We define the norm of $\mathfrak{a}=(a+ib)\mathbb{Z}[i]$ to be the norm of its generator: $(\mathfrak{a})=a^2+b^2$. Let χ_{-4} be the nonprincipal character modulo 4 (there is only one). As usual, $s=\sigma+it$.

(a) Explain why the number of ideal $\mathfrak{a}$ with $N(\mathfrak{a})\le x$ is $\frac{\pi}{4}x+O(\sqrt{x})$.

(b) For $\sigma>1$, let $\zeta_K(s)=\sum_{\mathfrak{a}}N(\mathfrak{a})^{-s}$ be the Dedekind zeta function of K. Show that

$$\zeta_K(s)=\zeta(s)L(s,\chi_{-4})$$

(c) Show that for any two ideals $\mathfrak{a},\mathfrak{b}\subset\mathbb{Z}[i]$ we have $N(\mathfrak{a}\mathfrak{b})=N(\mathfrak{a})N(\mathfrak{b})$.

(d) Since $\mathbb{Z}[i]$ is a PID, we have that ideals factor uniquely into prime ideals. Show that for $\sigma>1$ we have

$$\zeta_K(s)=\prod_{\mathfrak{p}}\left(1-\frac{1}{N(\mathfrak{p})}\right)^{-1}$$

where the product runs over the prime ideals.

(e) Define $\mu_K(\mathfrak{a})$ in such a way that

$$\frac{1}{\zeta_K(s)}=\sum_{\mathfrak{a}}\frac{\mu_K(\mathfrak{a})}{N(\mathfrak{a})^s}$$

Show that

$$\sum_{\mathfrak{d}|\mathfrak{a}\text{ and }\mathfrak{d}|\mathfrak{b}}\mu_K(\mathfrak{d})=\begin{cases}1 & \text{if } \gcd(\mathfrak{a},\mathfrak{b})=1\\ 0 & \text{otherwise}\end{cases}$$

(f) Consider the set of pairs of ideals $V(x)=\{\mathfrak{a}\subset\mathbb{Z}[i]:N(\mathfrak{a})\le x\}$. Show that the *probability* that two ideals $\mathfrak{a},\mathfrak{b}$ randomly drawn from $V(x)$ have $\gcd(\mathfrak{a},\mathfrak{b})=1$ is

$$\frac{1}{\zeta_K(2)}+O(x^{-1/2})=\frac{6}{\pi^2L(2,\chi_{-4})}+O(x^{-1/2})$$

(this uses that $\zeta(2)=\pi^2/6$, a famous result worth giving a presentation about, but you can assume it here)