Math 322 Spring 1996 Assignment 2: Priestly, Chapter 2

1. Let $f(z) := \mathrm{Re}(z)$. If $z = x + iy$, we can write $f(z) = f(x+iy) = u(x,y) + iv(x,y)$, where $u(x,y) := x$ and $v(x,y) := 0$. Then, the partial derivatives are

$$\frac{\partial u}{\partial x} = 1; \frac{\partial v}{\partial y} = 0; \frac{\partial u}{\partial y} = 0; \frac{\partial v}{\partial x} = 0.$$

Clearly, $\partial u/\partial x \neq \partial v/\partial y$ for any $z \in \mathbb{C}$. Therefore, by the Cauchy-Riemann equations, f is not differentiable at any $z \in \mathbb{C}$.

Let $g(z) := \bar{z}$. If $z = x + iy$, we can write $g(z) = g(x+iy) = u(x,y) + iv(x,y)$, where $u(x,y) := x$ and $v(x,y) := -y$. Then, the partial derivatives are

$$\frac{\partial u}{\partial x} = 1; \frac{\partial v}{\partial y} = -1; \frac{\partial u}{\partial y} = 0; \frac{\partial v}{\partial x} = 0.$$

Clearly, $\partial u/\partial x \neq \partial v/\partial y$ for any $z \in \mathbb{C}$. Therefore, by the Cauchy-Riemann equations, g is not differentiable at any $z \in \mathbb{C}$.

Let $k(z) := |z|^2$. If $z = x + iy$, we can write $k(z) = k(x+iy) = u(x,y) + iv(x,y)$, where $u(x,y) := x^2 + y^2$ and $v(x,y) := 0$. Then, the partial derivatives are

$$\frac{\partial u}{\partial x} = 2x; \frac{\partial v}{\partial y} = 0; \frac{\partial u}{\partial y} = 2y; \frac{\partial v}{\partial x} = 0.$$

In general, $\partial u/\partial x \neq \partial v/\partial y$. The Cauchy-Riemann equations do not hold anywhere except at the point $z = 0$, so k is clearly not differentiable at any point $z \in \mathbb{C}\backslash\{0\}$. At the point $z = 0$, we have

$$\lim_{h\to 0} \frac{k(0+h) - k(0)}{h} = \lim_{h\to 0} \frac{|h|^2}{h} = \lim_{h\to 0} \bar{h} = 0.$$

Therefore, k is differentiable at $z = 0$ and k is not differentiable at any $z \in \mathbb{C}\backslash\{0\}$.

The Cauchy-Riemann equations are *necessary* conditions for a complex function to be differentiable at $z \in \mathbb{C}$. They can be used to prove a function is *not* differentiable at $z \in \mathbb{C}$, but if even if they do hold at $z \in \mathbb{C}$, it is possible that the function $u + iv$ is still not differentiable there. Further, it is necessary to check $z = 0$ separately for the function k but not for f, g because the Cauchy-Riemann equations *fail* to hold for f, g at $z = 0$ while they do hold at $z = 0$ for k.

2. Let $f(z) := [\mathrm{Im}(z)]^2$. If $z = x + iy$, we can write $f(z) = f(x+iy) = u(x,y) + iv(x,y)$, where $u(x,y) := y^2$ and $v(x,y) := 0$. Then, the partial derivatives are

$$\frac{\partial u}{\partial x} = 0; \frac{\partial v}{\partial y} = 0; \frac{\partial u}{\partial y} = 2y; \frac{\partial v}{\partial x} = 0.$$

The Cauchy-Riemann equations fail to hold everywhere except on the real axis, ie. the set $S := \{z \in \mathbb{C} : \mathrm{Im}(z) = 0\}$. For $z \in S$, $z = x \in \mathbb{R}$. Thus, $\mathrm{Im}(z) = \mathrm{Im}(x) = 0$ and

$$\mathrm{Im}(z+h) = \mathrm{Im}(x+h) = \mathrm{Im}(h)$$

for any $h \in \mathbb{C}$. Hence, for $z \in S$, the derivative of f can be found from the definition:

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{[\mathrm{Im}(x+h)]^2 - [\mathrm{Im}(x)]^2}{h} = \lim_{h\to 0} \frac{[\mathrm{Im}(h)]^2}{h} = 0.$$

Thus, f is differentiable at any $z \in S$ and is non-differentiable everywhere else.

3. Define

$$f(z) := \begin{cases} 0 & , z = 0 \\ z^5/|z|^4 & , z \neq 0 \end{cases}.$$

If $z = x + iy$, then $f(z) = u(x,y) + iv(x,y)$ where

$$u(x,y) = \begin{cases} x\left(x^4 - 10x^2y^2 + 5y^4\right) / \left(x^2+y^2\right)^2 & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases},$$

$$v(x,y) = \begin{cases} y\left(y^4 - 10x^2y^2 + 5x^4\right) / \left(x^2+y^2\right)^2 & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}.$$

To show the Cauchy-Riemann equations hold at $z = 0$, start from the definition of the partial derivative of a real-valued function of two real variables:

$$\frac{\partial u}{\partial x}(0,0) := \lim_{h\to 0}\frac{u(h,0)-u(0,0)}{h} = \lim_{h\to 0}\frac{h^5/(h^2+0)^2}{h} = 1,$$
$$\frac{\partial v}{\partial y}(0,0) := \lim_{k\to 0}\frac{v(0,k)-v(0,0)}{h} = \lim_{h\to 0}\frac{k^5/(0+k^2)^2}{k} = 1,$$
$$\frac{\partial u}{\partial y}(0,0) := \lim_{k\to 0}\frac{u(0,k)-u(0,0)}{k} = \lim_{h\to 0}\frac{0/(0+k^2)^2}{k} = 0,$$
$$\frac{\partial v}{\partial x}(0,0) := \lim_{h\to 0}\frac{v(h,0)-v(0,0)}{h} = \lim_{h\to 0}\frac{0/(h^2+0)^2}{h} = 0.$$

Thus, the Cauchy-Riemann equations do hold at $z = 0$. To show f is nondifferentiable at $z = 0$, start from the definition of the derivative of f. Approaching $z = 0$ along the real axis, let $h \in \mathbb{R}$ be a small real number. Then,

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{h^5/|h|^4-0}{h} = 1.$$

Similarly, to approach $z = 0$ along the imaginary axis, let $k \in \mathbb{R}$ be a small real number. Then, ik is a small purely imaginary number and

$$\lim_{k\to 0}\frac{f(0+ik)-f(0)}{ik} = \lim_{k\to 0}\frac{(ik)^5/|ik|^4-0}{ik} = -1.$$

Thus, the limit does not exist. Hence, f is not differentiable at $z = 0$.

4. (iv) Consider the function $f(z) := z|z|$. If $z = x + iy$ and $f(z) = u(x,y) + iv(x,y)$, then, in this case

$$f(x+iy) = u(x,y) + iv(x,y)$$
$$u(x,y) = x\sqrt{x^2+y^2}$$
$$v(x,y) = y\sqrt{x^2+y^2}$$

The Cauchy-Riemann equations are given by

$$u_x - v_y = \frac{x^2-y^2}{\sqrt{x^2+y^2}} = 0$$
$$u_y + v_x = \frac{2xy}{\sqrt{x^2+y^2}} = 0$$

These equations should both be zero in any open set in which $z|z|$ is holomorphic. Since these equations are both zero *only* at $z = 0$, it follows that $z|z|$ is not holomorphic in $\mathbb{C}\backslash\{0\}$. Finally, since any open set including $z = 0$ includes points in $\mathbb{C}\backslash\{0\}$ where the derivative does not exist, it follows that $z|z|$ is not holomorphic anywhere in $\mathbb{C}$.

5. (iv) Consider the function $f(z) := \sum_{n=0}^{\infty}(n^n z^n)$. Applying the Root Test,

$$\lim_{n\to\infty}|n^n z^n|^{\frac{1}{n}} = n|z| \to +\infty$$

for any $z \in \mathbb{C}\backslash\{0\}$. For $z = 0$, the series obviously converges to zero. Thus, this function is nowhere holomorphic since the series does not converge in any open disk.

6. Let $< a_n >$ be a sequence of complex numbers such that $\sum_{n=0}^{\infty}|a_n| < +\infty$. We know that $\sum_{n=0}^{\infty} a_n$ converges because $\sum_{n=0}^{\infty}|a_n|$ does (because absolute convergence $\Rightarrow$ convergence). Let $s_n := \sum_{k=0}^{n} a_n$ and $\sigma_n := \sum_{k=0}^{n}|a_n|$. By the triangle inequality, $|s_n| \leq \sigma_n$. Hence,

$$\left|\sum_{n=0}^{\infty} a_n\right| = \left|\lim_{n\to\infty} s_n\right| \leq \lim_{n\to\infty}|s_n| \leq \lim_{n\to\infty}\sigma_n = \sum_{n=0}^{\infty}|a_n|.$$

(a) Now, using the fact that $e = \sum_{n=0}^{\infty}(1/n!)$, if $|z| \leq 1$,

$$0 \leq \sum_{n=2}^{\infty} \frac{|z|^{n-1}}{n!} \leq \sum_{n=2}^{\infty} \frac{1}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} - 2 = e - 2 \leq 1 \Rightarrow 1 - \sum_{n=2}^{\infty} \frac{|z|^{n-1}}{n!} \geq \dot{0}.$$

(b) Using the above results,

$$\begin{aligned}
(3-e)|z| = \left(3 - \sum_{n=0}^{\infty} \frac{1}{n!}\right)|z| &= 3|z| - |z|\left(\frac{1}{0!} + \frac{1}{1!} + \sum_{n=2}^{\infty} \frac{1}{n!}\right) \\
&= |z|\left(1 - \sum_{n=2}^{\infty} \frac{1}{n!}\right) \leq |z|\left(1 - \sum_{n=2}^{\infty} \frac{|z|^{n-1}}{n!}\right) \\
\leq |z|\left(1 - \left|\sum_{n=2}^{\infty} \frac{z^{n-1}}{n!}\right|\right) &= |z|\left(\left|1 + \sum_{n=2}^{\infty} \frac{z^{n-1}}{n!} - \sum_{n=2}^{\infty} \frac{z^{n-1}}{n!}\right| - \left|\sum_{n=2}^{\infty} \frac{z^{n-1}}{n!}\right|\right) \\
&\leq |z|\left|1 + \sum_{n=2}^{\infty} \frac{z^{n-1}}{n!}\right| = \left|z + \sum_{n=2}^{\infty} \frac{z^n}{n!}\right| = |e^z - 1|.
\end{aligned}$$

(c) Further, if $|z| \leq 1$, then

$$|e^z - 1| = \left|\sum_{n=1}^{\infty} \frac{z^n}{n!}\right| \leq \sum_{n=1}^{\infty} \left|\frac{z^n}{n!}\right| = |z| \sum_{n=1}^{\infty} \frac{|z|^{n-1}}{n!} \leq |z| \sum_{n=1}^{\infty} \frac{1}{n!} = (e-1)|z|.$$

Combining the above two inequalities, if $|z| \leq 1$,

$$(3-e)|z| \leq |e^z - 1| \leq (e-1)|z|.$$

7. Consider the series

$$\sum_{n=0}^{\infty} \frac{z^n}{1-z^n}$$

For $|z| \geq 1$, $1 + |z|^n \leq 2|z|^n$ for any (positive) integral power of $|z|$. Thus, for $|z| \geq 1$,

$$\left|\frac{z^n}{1-z^n}\right| = \frac{|z|^n}{|1-z^n|} \geq \frac{|z|^n}{1+|z^n|} = \frac{|z|^n}{1+|z|^n} \geq \frac{1}{2}.$$

Since the magnitude of the terms in the series do not tend to zero, this series cannot converge if $|z| \geq 1$. On the other hand, suppose $|z| < 1$; then, using the Ratio Test,

$$\lim_{n\to\infty} \left|\frac{z^{n+1}}{1-z^{n+1}} \cdot \frac{1-z^n}{z^n}\right| = |z| \lim_{n\to\infty} \left|\frac{1-z^n}{1-z^{n+1}}\right| = |z| \left|\frac{1-(0)}{1-(0)}\right| = |z| < 1$$

as $n \to \infty$. Therefore, the series is absolutely convergent only within the disk $D(0;1)$.

8. Consider the function $f(z) := 1/(z+1)(z+2)$. Then, using partial fractions,

$$\begin{aligned}
f(z) = \frac{1}{(z+1)(z+2)} &= \frac{1}{z+1} - \frac{1}{z+2} \\
&= \frac{1}{1-(-z)} - \frac{1}{2}\frac{1}{1-(-z/2)} \\
&= \sum_{n=0}^{\infty} (-1)^n z^n - \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n (z/2)^n \\
= \sum_{n=0}^{\infty} (-1)^n \left(1 - \frac{1}{2^{n+1}}\right) z^n &= \sum_{n=0}^{\infty} (-1)^n \left(\frac{2^{n+1}-1}{2^{n+1}}\right) z^n
\end{aligned}$$

(The coefficient of the nth term can also be found using the Maclaurin series expansion of f around $z = 0$). The first geometric series (in $(-z)$) converges for $|z| < 1$, while the second geometric series (in $(-z/2)$) converges for $|z| < 2$. The resulting sum of the two series converges only when $|z| < 1$.

9. Let $x := \mathrm{Re}(z)$ and $y := \mathrm{Im}(z)$. Then,

$$\begin{aligned} e^{e^z} = \exp\left(\exp(x+iy)\right) &= \exp(e^x(\cos y + i\sin y) \\ &= \exp\left(e^x \cos y + ie^x \sin y\right) \\ &= \exp(e^x\cos y)\cdot[\cos(e^x \sin y) + i\sin(e^x\sin y)] \end{aligned}$$

Thus, in terms of x and y, the real and imaginary parts of e^{e^z} are

$$\mathrm{Re}\left(e^{e^z}\right) = e^{e^x\cos y}\cos(e^x\sin y) \text{ and}$$
$$\mathrm{Im}\left(e^{e^z}\right) = e^{e^x\cos y}\sin(e^x\sin y).$$

10. Suppose that f is holomorphic in $D(0;1)$.

(ii) Suppose $e^f = C$, where C is a constant. Since f is holomorphic in $D(0;1)$ and the exponential function is holomorphic everywhere, the resulting composition e^f is also holomorphic in $D(0;1)$.The derivative of e^f exists in that domain and is found with the chain rule:

$$\frac{d}{dz}\left(e^{f(z)}\right) = f'(z)[e^{f(z)}] = \frac{d}{dz}(C) = 0$$

for every $z\in D(0;1)$. Since $e^w \neq 0$ for every $w\in\mathbb{C}$, it follows that $f'(z) = 0$ for $z\in D(0;1)$. It follows (using Proposition 2.6 (1)) that f is constant in $D(0;1)$.

11. (v) Consider $f(z) := \sin\left(\frac{1}{z^3}\right)$. Since the zeroes of $\sin z$ are the integral multiples of π on the real axis, it follows that the zeroes of f are the (real) cube roots of the integral multiples of π. That is,

$$\begin{aligned} f(z) = 0 \Rightarrow \sin\left(\frac{1}{z^3}\right) = 0 \Rightarrow \frac{1}{z^3} &= k\pi,\ (k\in\mathbb{Z}) \\ \Rightarrow z^3 = \frac{1}{k\pi} &\in \mathbb{R},\ (k\in\mathbb{Z}) \\ \Rightarrow z = (k\pi)^{-\frac{1}{3}}&,\ (k\in\mathbb{Z}). \end{aligned}$$

Thus, $S := \{z\in\mathbb{C} : f(z) = 0\} = \{z = (k\pi)^{-\frac{1}{3}} \in\mathbb{R} : k\in\mathbb{Z}\}$.

12. (iv)

$$\begin{aligned} \tan z = i \Rightarrow \sin z &= i\cos z \\ \Rightarrow (e^z - e^{-z})/2i &= i(e^z + e^{-z})/2 \\ \Rightarrow e^z - e^{-z} &= -e^z - e^{-z} \\ \Rightarrow e^z &= 0. \end{aligned}$$

Since $e^z \neq 0$ everywhere in $\mathbb{C}$, it follows that this equation has no solutions; that is,

$$\{z\in\mathbb{C} : \tan z = i\} = \{\} = \emptyset.$$

13. (ii) Consider the mapping $z\mapsto e^z$. If we write $z = a+bi$, where $a, b\in\mathbb{R}$, it follows that $e^z = e^{a+bi} = e^a e^{bi}$. The modulus of this complex number is $e^a \in\mathbb{R}$ and its argument is b.Thus, the horizontal line $A := \{z\in\mathbb{C} : \mathrm{Im}(z) = b\}$ in the complex plane is mapped to the radial half-line $B := \{z\in\mathbb{C} : z = r(\cos b + i\sin b) \text{ for some } r > 0\}$ extending from the origin that makes an angle b (in radian measure) with the real axis.

14. (ii) Evaluate the following complex logarithms:

$$\begin{aligned} [\log(-1)] &= \{\log|-1| + i\theta : \theta\in[\arg(-1)]\} = \{0 + i(-\pi + 2n\pi) : n\in\mathbb{Z}\} \\ &= \{(2n-1)\pi i : n\in\mathbb{Z}\}; \\ [\log(1+i)] &= \{\log|1+i| + i\theta : \theta\in[\arg(1+i)]\} = \left\{\log\left(\sqrt{2}\right) + i(\pi/4 + 2n\pi) : n\in\mathbb{Z}\right\} \\ &= \left\{\frac{1}{2}\log 2 + \frac{1}{4}(8n+1)\pi i : n\in\mathbb{Z}\right\}; \\ [\log e^z] &= \{\log|e^z| + i\theta : \theta\in[\arg(e^z)]\} = \{\log\left|e^{\mathrm{Re}\, z}\right| + i\theta : \theta\in[\arg(e^{(\mathrm{Im}\, z)i})]\} \\ &= \{\mathrm{Re}(z) + i(\mathrm{Im}(z) + 2n\pi) : n\in\mathbb{Z}\}. \end{aligned}$$

15. Find all the roots of the following complex numbers:

$$\left[1^{\frac{1}{3}}\right] = \{z = e^{k\pi i/3} : k = 0, 1, 2\} = \left\{1, -\frac{1}{2} + \frac{i}{2}\sqrt{3}, -\frac{1}{2} - \frac{i}{2}\sqrt{3}\right\};$$

$$\left[(-8)^{\frac{1}{3}}\right] = \left\{z = -(8)^{\frac{1}{3}} e^{k\pi i/3} : k = 0, 1, 2\right\} = \left\{-2, 1 + i\sqrt{3}, 1 - i\sqrt{3}\right\};$$

$$\left[i^{\frac{1}{6}}\right] = \left\{z = \left(e^{\left(2k+\frac{1}{2}\right)\pi i}\right)^{\frac{1}{6}} : k = 0, 1, 2, 3, 4, 5\right\}$$

$$= \left\{e^{\pi i/12}, e^{5\pi i/12}, e^{9\pi i/12}, e^{13\pi i/12}, e^{17\pi i/12}, e^{21\pi i/12}\right\};$$

$$\left[(-1)^{\frac{1}{4}}\right] = \left\{e^{(-1+2n)\pi i/4} : n = 0, 1, 2, 3\right\} = \left\{\frac{1-i}{\sqrt{2}}, \frac{1+i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}\right\}.$$

16. Evaluate the following complex powers:

$$\left[\sqrt{2}^{i}\right] = \left\{e^{\frac{i}{2}(\log|2|+i\theta)} : \theta \in [\arg 2]\right\} = \left\{e^{\frac{i}{2}(\log 2 - 2n\pi)} : n \in \mathbb{Z}\right\}$$

$$= \left\{e^{n\pi + (\log 2)i/2} : n \in \mathbb{Z}\right\}$$

(Note: $\sqrt{2^i} = \sqrt{2}^i$ since we can regard both as being $2^{(i/2)}$).

$$\left[i^{\sqrt{2}}\right] = \left\{e^{\sqrt{2}(\log|i|+i\theta)} : \theta \in [\arg(i)]\right\} = \left\{e^{\sqrt{2}(1/2+2n)\pi i} : n \in \mathbb{Z}\right\}$$

$$= \left\{e^{(4n+1)\sqrt{2}\pi i/2} : n \in \mathbb{Z}\right\}.$$

Thus, $i^{\sqrt{2}}$ is an infinite set of points on the unit circle.

$$\left[i^{i}\right] = \left\{e^{i(\log|i|+i\theta)} : \theta \in [\arg(i)]\right\} = \left\{e^{i(0+(\pi/2+2n\pi)i)} : n \in \mathbb{Z}\right\}$$

$$= \left\{e^{(-4n-1)\pi/2} : n \in \mathbb{Z}\right\} = \left\{..., e^{-9\pi/2}, e^{-5\pi/2}, e^{-\pi/2}, e^{3\pi/2}, e^{7\pi/2}, ...\right\}$$

Hence, i^i is an infinite set of points on the positive half of the real axis.

$$\left[e^{i\pi}\right] = \left\{e^{i\pi(\log e+i\theta)} : \theta \in [\arg(e)]\right\} = \left\{e^{i\pi+2n\pi i} : n \in \mathbb{Z}\right\}$$

$$= \left\{e^{(2n+1)\pi i} : n \in \mathbb{Z}\right\} = \{-1\}.$$

It makes sense that $[e^{i\pi}]$ consists of only one value; $f(z) = e^z$ is meant to be a single-valued function.