Math 322: Assignment 3 Key

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1. Describe the image γ^* of the curve γ in the following cases:

   (i) $\gamma(t) = 1 + ie^{it}$ $(t \in [0, \pi])$.
   The image is a semi-circle of radius 1 centred at $z = 1$ in $\mathbb{C}$. The arc starts at $z = 1 + i$ and extends in a counterclockwise orientation to $z = 1 - i$. It is a simple, smooth path but is not closed.

   (ii) $\gamma(t) = e^{it}$ $(t \in [-\pi, 2\pi])$
   The image of this curve is a circle of radius 1 centred at the origin. It is a smooth path. It is not closed since $\gamma(-\pi) = -1 \neq 1 = \gamma(2\pi)$. It is not simple since the mapping is not one-to-one (ie. $\gamma(-\pi) = -1 = \gamma(\pi)$).

   (iii) γ is the join of [-1,1], [1,1 + i] and[$1 + i, -1 - i$].
   This set of points consists of a horizontal line segment of length 2 extending from $z = -1$ to $z = 1$, a vertical line segment extending from $z = 1$ to $z = 1+i$ and finally a line segment of slope 1 extending from $z = 1 + i$ to $z = -1 - i$. γ is not closed, nor is it simple since the first line segment crosses the third at the origin. It is a path that is piecewise smooth on each of the segments.

   (iv) γ is the join of γ_1, γ_2 and γ_3, where γ_1 is $[1 - i, 0]$, γ_2 is [0,1 + i] and γ_3 is defined by $\gamma(t) = \sqrt{2}e^{i(t+\pi/4)}$ $(t \in [0, 3\pi/3])$.
   This set of points is consists of the straight line segment of length $\sqrt{2}$ from $z = 1 - i$ to $z = 0$ joined in a right angle at the origin to the straight line segment of length $\sqrt{2}$ from $z = 0$ to $z = 1 + i$. From there, γ_3^* is the portion of the circular arc of radius $\sqrt{2}$ centred at $z = 0$ travelling counterclockwise from $z = 1 + i$ to $z = 1 - i$. This is a closed, simple path that is piecewise smooth. (It fails to be smooth at the corners $z = 1 - i$, $z = 0$ and $z = 1 + i$).

   (v) γ is given by $\gamma(t) = e^{it}$ $(t \in [0, \pi])$ and $\gamma(t) = e^{-it}$ $(t \in [\pi, 2\pi])$.
   As t varies from 0 to π, γ traces out the semicircle of radius 1 centred at $z = 0$ from $z = 1$ to $z = -1$ with counterclockwise orientation. As t varies from π to 2π, the same arc is traced out again but with clockwise orientation. Thus, γ is a closed piecewise smooth path. It is not simple and fails to be smooth at $t = \pi$.

(vi) $\gamma = e^{it}\cos t$ $(t \in [0, 2\pi])$.

γ traces out the image of a circle of radius $1/2$ centred at $z = 1/2$. It winds twice around the circle in a counterclockwise way. Thus, it is a closed, smooth path but is not simple.

3. Evaluate the following integrals:

(i) Let $\gamma(t) := 2t - 1 + i$, where $t \in [0, 1]$. Thus, γ traces out the line segment $[-1 + i, 1 + i]$ and $\gamma'(t) = 2$. If $f(z) = |z|^4$, then

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_0^1 f(\gamma(t))\gamma'(t)dt = \int_0^1 |2t - 1 + i|^4 (2)dt \\
&= 2\int_0^1 [(2t-1)^2 + (1)^2]^2 dt = 2\int_0^1 (4t^2 - 4t + 2)^2 dt \\
&= 2\int_0^1 (16t^4 - 32t^3 + 32t^2 - 16t + 4)dt \\
&= 2\left[\frac{16}{5}t^5 - \frac{32}{4}t^4 + \frac{32}{3}t^3 - \frac{16}{2}t^2 + 4t\right]_0^1 = \frac{56}{15}.
\end{aligned}$$

(ii) Let $\gamma(t) := e^{it}$, where $t \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and let $f(z) = z^2$. Since $F(z) = z^3/3$ is holomorphic everywhere in $\mathbb{C}$ and $F'(z) = f(z)$, we can invoke the Fundamental Theorem of Calculus (Theorem 3.8) to save some work. In particular, since F is continuous on an open set containing γ^* (say, all of $\mathbb{C}$),

$$\int_\gamma f(z)dz = \int_\gamma F'(z)dz = F\left(\gamma\left(\frac{\pi}{2}\right)\right) - F\left(\gamma\left(-\frac{\pi}{2}\right)\right) = \frac{1}{3}[(i)^3 - (-i)^3] = -\frac{2}{3}i.$$

(iii) Let $\gamma(t) := t + it^2$ where $t \in [0, 1]$ and let $f(z) := \mathrm{Re(z)}$. Then, $\gamma'(t) = 1 + 2ti$ and so

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_0^1 f(\gamma(t))\gamma'(t)dt = \int_0^1 \mathrm{Re(t + it^2)(1 + 2ti)dt} \\
&= \int_0^1 t(i + 2ti)dt = \int_0^1 tdt + 2i\int_0^1 t^2 dt \\
&= 2\left[\frac{1}{2}t^2\right]_0^1 + 2i\left[\frac{1}{3}t^3\right]_0^1 = \frac{1}{2} + \frac{2}{3}i.
\end{aligned}$$

(iv) Let $\gamma(t) := e^{-it}$ where $t \in [0, 8\pi]$ and let $f(z) := 1/z$. Now, $\gamma'(t) = -ie^{-it}$, so

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_0^1 f(\gamma(t))\gamma'(t)dt = \int_0^{8\pi} \left(\frac{1}{e^{-it}}\right)(-ie^{-it})dt \\
&= -i\int_0^{8\pi} e^{-it-(-it)}dt = -8\pi i.
\end{aligned}$$

(v) Let γ be the join of $[0,1]$, $[1,1+i]$ and $[1+i,i]$. Let $f(z) := e^z$. Once again, the Theorem 3.8 can be used to simplify the calculation. Take $F(z) := e^z$, so $F'(z) = f(z)$ for every $z \in \mathbb{C}$. Since F is continuous and differentiable in every open set (in particular, in any open set covering γ), we have

$$\int_\gamma f(z)dz = \int_\gamma F'(z)dz = F(i) - F(0) = e^i - e^0 = e^i - 1.$$

7. Let $f(z) := \sum_{n=0}^{\infty} c_n z^n$ have radius of convergence $R > 0$. Consider $g(z) := f(z)/z^{k+1}$, where k is a positive integer; this can be expressed by

$$g(z) = \frac{f(z)}{z^{k+1}} = \sum_{n=0}^{k-1} \frac{c_n}{z^{k+1-n}} + \frac{c_k}{z} + \sum_{n=0}^{\infty} c_{n+k+1} z^n.$$

To integrate $g(z)$, it is necessary to determine whether or not the series $\sum_{n=0}^{\infty} c_{n+k+1} z^n$ converges to justify integration term-by-term. Fixing $z \in D'(0;R)$, the series $f(z)$ converges absolutely there; thus, by the Cauchy convergence criterion, given any $\epsilon > 0$, there exists $m_0 \in \mathbb{N}$ such that, for $n_2 > n_1 > m_0$,

$$\sum_{n=n_1+1}^{n_2} |c_n z^n| < \epsilon |z|^{k+1}.$$

Then, given $\epsilon > 0$, choose $m' := m_0 - k - 1$; thus, for any $n_2 > n_1 > m'$,

$$\begin{aligned}
\sum_{n=n_1+1}^{n_2} |c_{n+k+1} z^n| &= \left|\frac{1}{z^{k+1}}\right| \sum_{n=n_1+1}^{n_2} |c_{n+k+1} z^{n+k+1}| \\
&= \left|\frac{1}{z^{k+1}}\right| \sum_{n=(n_1+k+1)+1}^{n_2+k+1} |c_n z^n| < \frac{1}{|z|^{k+1}} \cdot \epsilon |z|^{k+1} = \epsilon.
\end{aligned}$$

Thus, absolute convergence of the series for $g(z)$ follows from the Cauchy convergence criterion also. Thus, fix $r > 0$ such that $0 < r < R$. The series for f and g both converge absolutely on the circle $|z| = r$ in the complex plane. In particular, define $M_n := |c_n| r^{n-k}$ for $n > k$; at any point on $\gamma(0;r)$, $|c_n z^{n-k}| < M_n$ and $\sum M_n$ converges. By the Theorem 3.13, integration of $g(z)$ term-by-term along $\gamma(0;r)$ is justified. Thus,

$$\begin{aligned}
\int_{\gamma(0;r)} \frac{f(z)}{z^{k+1}} dz &= \int_0^{2\pi} \left(\sum_{n=0}^{k-1} \frac{c_n}{(re^{i\theta})^{k+1-n}} + \frac{c_k}{re^{i\theta}} + \sum_{n=0}^{\infty} c_{n+k+1} (re^{i\theta})^n \right) (ire^{i\theta} d\theta) \\
&= \sum_{n=0}^{k-1} i c_n r^{n-k} \int_0^{2\pi} e^{i\theta(n-k)} d\theta + i \int_0^{2\pi} c_k \, d\theta \\
&\quad + i \sum_{n=0}^{\infty} c_{n+k+1} r^{n+1} \int_0^{2\pi} e^{i\theta(n+1)} d\theta \\
&= 0 + 2\pi i c_k + 0.
\end{aligned}$$

Thus,

$$\frac{1}{2\pi i}\int_{\gamma(0;r)}\frac{f(z)}{z^{k+1}}dz = c_k.$$

Now, let $M := \sup\{|f(z)| : |z| = r\}$ be the least upper bound on the modulus of $f(z)$ on the circle of radius $r < R$ centred at the origin. Using the above result,

$$\begin{aligned}|c_k|r^k &= \frac{r^k}{2\pi}\left|\int_{\gamma(0;r)}\frac{f(z)}{z^{k+1}}dz\right| \le \frac{r^k}{2\pi}\int_0^{2\pi}\left|\frac{f(re^{i\theta})}{(re^{i\theta})^{k+1}}\cdot ire^{i\theta}\right|d\theta \\ &\le \frac{r^k}{2\pi}\int_0^{2\pi}\frac{M}{r^k}d\theta = M.\end{aligned}$$

Hence, $|c_k|r^k \le \sup\{|f(z)| : |z| = r\}$.

9. (i) Let G_1 and G_2 be regions. If $G := G_1 \cup G_2$ is a region, then $G_1 \cap G_2 \neq \emptyset$ by definition of connectedness. Conversely, suppose $G_1 \cap G_2 \neq \emptyset$. Let $z_0 \in G_1 \cap G_2 \neq \emptyset$. Pick two points $a, b \in G := G_1 \cup G_2$. If both points lie in G_1, since G_1 is a region, there is a polygonal path lying entirely in $G_1 \subset G_1 \cup G_2$ with endpoints a and b. If both points lie in G_2, since G_2 is a region, there is a polygonal path lying entirely in $G_2 \subset G_1 \cup G_2$ with endpoints a and b. If a lies in G_1 and b lies in G_2, then, there exists a path γ_1 lying in G_1 with initial point z and terminal point z_0. There is also a path γ_2 lying in G_2 with initial point z_0 and terminal point b. Then, the join of γ_1 and γ_2 is a polygonal path lying in $G_1 \cup G_2$ with endpoints a and b. Thus, for any pair of points in $G_1 \cup G_2$, there exists a polygonal path lying in $G_1 \cup G_2$ with endpoints a and b; it follows that $G_1 \cup G_2$ is polygonally connected and hence is connected. Thus, $G_1 \cup G_2$ is a region (since it is obviously open and nonempty)

   (ii) Let G be a region. Fix $z_1 \in G$. Consider $a, b \in G\backslash\{z_1\}$. Since $a, b \in G$ and G is a region, it follows that there exists a polygonal path γ lying in G with endpoints a and b. There are two possibilities:

   i. $z_1 \notin \gamma^*$. In this case, γ is a curve lying in $G\backslash\{z_1\}$ with endpoints a and b.

   ii. $z_1 \in \gamma^*$. Using the Covering Theorem, γ^* can be covered by a finite number of disks $D_0, \ldots, D_N$. Since each of these disks is a convex set, there exists an elementary deformation from γ to some other curve $\tilde{\gamma}$ that avoids z_1 and has endpoints a and b (if $z_1 \in D_m$ then replace $\gamma^* \cap D_m$ with a portion of circular arc within D_m that avoids z_1).

   In either case, there is a path lying entirely in $G\backslash\{z_1\}$ with endpoints a and b; hence $G\backslash\{z_1\}$ is connected. Since $G\backslash\{z_1\}$ is obviously open and nonempty, it follows that $G\backslash\{z_1\}$ is a region. Since the result of removing a single point from a connected region is still a region, it follows by induction that removing any finite set of points from G cannot make G disconnected. Thus, $G\backslash\{z_1, z_2, \ldots, z_n\}$ is a region.

10. Note: although Priestly restricts the definition of simple-connectedness to open, connected regions, below I consider a set simply-connected if any closed curve lying wholely within the region is homotopic to a point in the region. Thus, even if a set is not open or connected, it may be simply-connected.

    (i) $S = \{z \in \mathbb{C} : |z - 1 - i| = 1\}$ is a circle. Being a curved one-dimensional subset of $\mathbb{C}$, it's not convex, not a region (since it's not open) and not simply-connected.

    (ii) $S = \{z \in \mathbb{C} : |z - 1 + i| \geq |z - 1 - i|\}$ is the closed upper half-plane; it's convex, not a region (because it's not open) and but it is simply-connected (in the sense above).

    (iii) $S = \{z \in \mathbb{C} : |z + i| \neq |z - i|\}$ is all of $\mathbb{C}$ except the real axis. S is not convex (since, any line segment crossing the real axis, such as $[1+i, 1-i]$, is not contained in S). S is not a region since $S = S_1 \cup S_2$, where $S_1 := \{z \in \mathbb{C} : \mathrm{Im(z)} > 0\}$ and $S_2 := \{z \in \mathbb{C} : \mathrm{Im(z)} < 0\}$ and $S_1 \cap S_2 = \emptyset$ It is simply connected (any closed curve in S lies entirely in S_1 or S_2 and is homotopic to a point in either).

    (iv) $S = \{z = |z|e^{i\phi} \in \mathbb{C} : \frac{\pi}{4} < \phi \leq \frac{3\pi}{4}\}$ is a wedge with a corner at $z = 0$. S is convex, but it is not open. It's not a region, but it is simply-connected (in the sense above).

    (v) $S = \{z \in \mathbb{C} : \mathrm{Re(z)} < 1 \text{ or } \mathrm{Im(z - 1)} \neq 0\}$ is not convex, is a region, and is simply-connected.

    (vi) $S = \{z \in \mathbb{C} : |z - 1| < 1, |z| = |z - 2|\}$ is a line segment; it is convex, it is not a region and it is simply-connected since any closed curve lying in S is a smaller line segment homotopic to a point on the line.

    (vii) $S = \{z \in \mathbb{C} : |z - 2| < 3, |z| < 2\}$ is a crescent-shaped region. It is not convex, a region and simply-connected.

    (viii) $S = \{z \in \mathbb{C} : |z^2 - 1| < 1\}$ is the region inside a lemniscate. it is not convex and not a region (since $z = 0$ does not lie in S) although it is simply-connected in either half.

    (ix) $S = \{z \in \mathbb{C} : |z|^2 > z + \overline{z}\}$ is all of $\mathbb{C}$ with a closed disk removed. It is not convex, a region, but not simply-connected.

    (x) $S = \{z \in \mathbb{C} : \mathrm{Im((z + i)/2i)} < 0\}$ is the right half-plane where $\mathrm{Im(z)} > 0$. It is convex, a region and simply-connected.