Math 322: Assignment 4 Key

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1. - $\int_\gamma (1/(z-2)dz = 0$ where $\gamma^* \subset D(0;1)$ because $f(z) := 1/(z-2)$ is holomorphic throughout the entire disk $D(0;1)$. Thus, the result follows from Cauchy's Theorem for such a closed curve γ.
   - $\int_{\gamma(i;3)} (1/(z-2)^3)dz = 0$ since, by the Deformation theorem and the example 3.7 (p. 34),
   $$\int_{\gamma(i;3)} \frac{dz}{(z-2)^3} = \int_{\gamma(2;1)} \frac{dz}{(z-2)^3} = 0.$$
   - $\int_{\gamma(1;2)} (\sin z/z)dz = 0$ since $f(z) := \sin z/z$ has a removable discontinuity at $z = 0$ so f is in fact holomorphic throughout the interior of $\gamma(1;2)$.
   - $\int_{\gamma(0;1)} z|z|^4 dz = \int_0^{2\pi} e^{it}|e^{it}|^4(ie^{it}dt) = i\int_0^{2\pi} e^{2it}dt = 0.$
   - $\int_{\gamma(1;1)} (1+e^z)^{-1}dz = 0$ because $f(z) := (1+e^z)^{-1}$ is holomorphic in $\mathbb{C}\backslash\{2k\pi i : k \in \mathbb{Z}\}$. In particular, f is holomorphic on an open, simply connected region that includes $\gamma^* \cup I(\gamma)$; by Cauchy's theorem, the integral vanishes.

2. Let $f(z) := (1+z^2)^{-1}$. Then, f can be decomposed by partial fractions into
$$f(z) = \frac{i}{2}\left(\frac{1}{z+i} - \frac{1}{z-i}\right).$$
If γ is a closed circle in the complex plane, the integral of f around γ depends on whether the singularities $z = \pm i$ lie within $I(\gamma)$. There are 3 possibilities, all of which rely on the Deformation theorem.
   - if $i \in I(\gamma)$ and $-i \notin I(\gamma)$, then γ can be deformed into a circle $\gamma(i;\epsilon)$ for ϵ sufficiently small. In this case, $\int_\gamma f(z)dz = (i/2)(0 - 2\pi i) = \pi$.
   - if $-i \in I(\gamma)$ and $i \notin I(\gamma)$, then γ can be deformed into a circle $\gamma(-i;\epsilon)$ for ϵ sufficiently small. In this case, $\int_\gamma f(z)dz = (i/2)(2\pi i - 0) = -\pi$.
   - If both i and $-i$ lie in $I(\gamma)$ or $I(\gamma)$ avoids both, $\int_\gamma f(z)dz = 0$. In the former case, γ can be deformed into the disks $\gamma(i;\epsilon)$ and $\gamma(-i;\epsilon)$ so the integral evaluates to $\pi + (-\pi) = 0$. In the latter case, Cauchy's theorem applies.

(All of these cases assume that $\pm i \notin \gamma^*$.) The evaluation of $\int_\gamma f(z)dz$ around the following contours becomes trivial:

(i) $\gamma(0;1)$: $\int_\gamma f(z)dz = 0$ since $i, -i \notin D(0;1)$.

(ii) $\gamma(i;1)$: $\int_\gamma f(z)dz = \pi$ since $i \in D(i;1), -i \notin D(0;1)$.

(iii) $\gamma(-i;1)$: $\int_\gamma f(z)dz = -\pi$ since $-i \in D(-i;1), i \notin D(-i;1)$.

(iv) $\gamma(0;2)$: $\int_\gamma f(z)dz = 0$ since $i, -i \in D(0;2)$.

(v) $\gamma(3i;\pi)$: $\int_\gamma f(z)dz = \pi$ since $i \in D(i;1), -i \notin D(0;1)$.

3. Let f be holomorphic on a regular polygon parametrised by γ and within its interior. If the nodes of the polygon are $z_1, z_2, \dots, z_n$, and the centre of the polygon is z_0, then then the polygon can be subdivided into n triangles $\gamma_1 := [z_0, z_1, z_2]$, $\gamma_2 := [z_0, z_2, z_3], \dots \gamma_n := [z_0, z_n, z_1]$ (using the notation of the proof for Theorem 4.2). Thus, using Theorem 4.2,

$$\int_\gamma f(z)dz = \sum_{k=1}^{n} \int_{\gamma_k} f(z)dz = 0.$$

4. Let γ be a polygonal path with initial point $z = 0$ and final point $z = 1$.

(i) If $f(z) = z^3$, then f is holomorphic everywhere and has an antiderivative $F(z) := z^4/4$ that is also holomorphic everywhere in $\mathbb{C}$. Thus, the Fundamental Theorem of Calculus applies and the value of the integral is independent of the specific path γ:

$$\int_\gamma z^3 dz = (1^4 - 0^4)/4 = 1/4.$$

(ii) If $f(z) = (1 + z^2)^{-1}$, then f has an antiderivative $F(z) := \arctan(z)$ along the real axis. The polygonal path γ can be deformed into the join of $\gamma(i;1)$, $\gamma(-i;1)$ and [0,1], where γ may wind round each of these disks some integral number of times (positive or negative, depending on orientation). The analysis in question 4.2 for this function tells us that the integral of f around some curve γ increases by $+\pi$ each time γ winds a closed path around i and increases by $-\pi$ each time γ winds around $-i$. The Fundamental theorem of Calculus tells us that $\int_{[0,1]} f(z)dz = \pi/4$. Thus,

$$\int_\gamma (1 + z^2)^{-1} dz = \left(m + \frac{1}{4}\right)\pi, \quad \text{where } m \in \mathbb{Z}.$$

5. Let $\gamma(t) := a\cos t + ib\sin t$ $(t \in [0, 2\pi])$ be a parametric description of an ellipse in the complex plane. Let $f(z) := z^{-1}$; then f is holomorphic in any open annulus that includes γ^* and excludes the origin. Since γ

is homotopic to the curve $\gamma(0;\epsilon)$, where $\epsilon > 0$, and γ is oriented anti-clockwise, the Deformation Theorem 4.11 asserts that

$$\int_{\gamma(0;\epsilon)} \frac{dz}{z} = \int_\gamma \frac{dz}{z}.$$

We know the left-hand integral is $2\pi i$ (by Example 3.7, p. 34); thus, it follows that

$$\begin{aligned}
2\pi &= \operatorname{Im}\left(\int_\gamma \frac{dz}{z}\right) = \operatorname{Im}\left(\int_0^{2\pi} \frac{\gamma'(t)}{\gamma(t)} dt\right) \\
&= \operatorname{Im}\left(\int_0^{2\pi} \frac{-a\sin t + ib\cos t}{a\cos t + ib\sin t} dt\right) \\
&= \operatorname{Im}\left(\int_0^{2\pi} \frac{(-a\sin t + ib\cos t)(a\cos t - ib\sin t)}{(a\cos t + ib\sin t)(a\cos t - ib\sin t)} dt\right) \\
&= \operatorname{Im}\left(i\int_0^{2\pi} \frac{ab(\sin^2 t + \cos^2 t)}{a^2\cos^2 t + b^2\sin^2 t} dt + \int_0^{2\pi} \frac{(b^2 - a^2)\sin t\cos t}{a^2\cos^2 t + b^2\sin^2 t} dt\right) \\
&= ab\int_0^{2\pi} \frac{dt}{a^2\cos^2 t + b^2\sin^2 t}.
\end{aligned}$$

Thus, it follows that

$$\int_0^{2\pi} \frac{dt}{a^2\cos^2 t + b^2\sin^2 t} = \frac{2\pi}{ab}.$$

6. Let $\mathbb{C}_\pi := \{z = |z|e^{i\theta} \neq 0 : -\pi < \theta < \pi\}$. Let f be defined by $f(z) := \int_\gamma (1/w)dw$, where $z = |z|e^{i\theta} \in \mathbb{C}_\pi$ and γ is the join of $\gamma_1 := [1, |z|]$ and γ_2 defined by

$$\gamma_2(t) := \begin{cases} |z|e^{it} & (t \in [0,\theta]) \text{ if } \operatorname{Im}(z) \geq 0, \\ |z|e^{i(\theta - t)} & (t \in [\theta, 0]) \text{ if } \operatorname{Im}(z) < 0. \end{cases}$$

Thus, along γ_1, w is real, so the integral $\int_{\gamma_1}(1/w)dw$ can be treated as an integral from calculus of a single real variable:

$$\int_{\gamma_1} \frac{dw}{w} = \int_1^{|z|} \frac{dt}{t} = (\ln|t|)\Big|_{t=1}^{t=|z|} = \ln|z|.$$

Along γ_2, the result of the integral breaks down into 2 cases.

- If $z \in \mathbb{C}_\pi$ and $\operatorname{Im}(z) \geq 0$,

$$\int_{\gamma_2} \frac{dw}{w} = \int_0^\theta \frac{i|z|e^{it}}{|z|e^{it}} dt = i\theta.$$

- If $z \in \mathbb{C}_\pi$ and $\mathrm{Im}(z) < 0$,

$$\int_{\gamma_2} \frac{dw}{w} = \int_\theta^0 \frac{i|z|e^{i(\theta-t)}}{|z|e^{i(\theta-t)}} dt = i\theta.$$

In both cases,

$$f(z) = \int_\gamma \frac{dw}{w} = \int_{\gamma_1} \frac{dw}{w} + \int_{\gamma_2} \frac{dw}{w} = \ln|z| + i\theta.$$

7. Let G be an open set and let $f \in H(G)$ satisfy $f = e^g$ for some $g \in H(G)$. For any $n \geq 2$, the function h defined by $h(z) := e^{g(z)/n}$ is holomorphic since it is the composition of two holomorphic functions. Further,

$$[h(z)]^n = [e^{g(z)/n}]^n = e^{g(z)} = f(z).$$

Hence, for any $n \geq 2$, there is a function $h \in H(G)$ such that $h^n = f$.

8. Suppose θ is a continuous function in $\mathbb{C}\backslash\{0\}$ with the property that $\theta(z) \in [\arg z]$ for every $z \neq 0$. Consider $\gamma(t) := e^{it}$, where $t \in [-\pi, \pi]$. The function γ is continuous on the compact set $[-\pi, \pi]$ and, by assumption, θ is continuous on γ^*. Hence, the function k defined by

$$k(t) := 1/(2\pi)(\theta(e^{it}) + \theta(e^{-it}))$$

is a continuous real-valued function on the compact interval $[-\pi, \pi]$. For any $t \in [-\pi, \pi]$, by assumption,

$$\begin{aligned} \theta(e^{it}) \in [\arg e^{it}] &= \{t + 2m\pi : m \in \mathbb{Z}\} \text{ and} \\ \theta(e^{-it}) \in [\arg e^{-it}] &= \{-t + 2n\pi : n \in \mathbb{Z}\}. \end{aligned}$$

Hence, there exists some $m, n \in \mathbb{Z}$ such that

$$k(t) = \frac{1}{2\pi}((t + 2m\pi) + (-t + 2n\pi)) = m + n \in \mathbb{Z}.$$

Since k is integer-valued and continuous on a compact interval, it follows from 1.19 that k is constant valued. Thus, for any $t \in [-\pi, \pi]$, $k(t) = m_0$ for some fixed $m_0 \in \mathbb{Z}$. Hence,

$$2\pi m_0 = 2\pi k(0) = 2\theta(1) \text{ and } 2\pi m_0 = 2\pi k(\pi) = 2\theta(-1).$$

This implies that $\theta(1) = \theta(-1)$; however, by assumption, $\theta(1) \in [\arg 1]$ and $\theta(1) = \theta(-1) \in [\arg(-1)]$ which is not possible as $[\arg 1] \cap [\arg(-1)] = \emptyset$. By contradiction, no such function θ exists.