Math 322: Assignment 5 Key

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Priestly, Chapter 6.

1. Find the Laurent series expansion for $f(z) = (z^2-1)^{-2}$ that is valid in the appropriate domains.

   (i) $0 < |z-1| < 2$; write $f(z)$ as

$$f(z) = \frac{1}{(z^2-1)^2} = \frac{1}{(z-1)^2} \cdot \frac{1}{((z-1)+2)^2} \tag{1}$$

Now, since $((z-1)+2)^{-2} = -\frac{d}{dz}((z-1)+2)^{-1}$, it follows that

$$\begin{aligned}
\frac{1}{((z-1)+2)^2} &= -\frac{d}{dz}\left(\frac{1}{((z-1)+2)}\right) = -\frac{1}{2}\frac{d}{dz}\left(\frac{1}{1+\frac{z-1}{2}}\right) \\
&= -\frac{1}{2}\frac{d}{dz}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2)^n}(z-1)^n\right) \\
&= -\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n n}{(2)^n}(z-1)^{n-1} \\
&= \sum_{n=1}^{\infty}\frac{(-1)^{n+1} n}{2^{n+1}}(z-1)^{n-1}. \qquad (2)
\end{aligned}$$

The power series expansion in (2) is valid provided $0 \le |z-1| < 2$; the series also converges uniformly which justifies differentiation term by term. Thus, substituting the series in (2) into the expression for $f(z)$ in (1) gives

$$f(z) = \frac{1}{(z-1)^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{2^{n+1}}(z-1)^{n-1} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{2^{n+1}}(z-1)^{n-3}.$$

It follows, then, that

$$f(z) = \sum_{n=-2}^{\infty}\frac{(-1)^n(n+3)}{2^{n+4}}(z-1)^n \tag{3}$$

is the Laurent series expansion in the region $0 < |z-1| < 2$.

(ii) If $|z+1|>2$, the Laurent series expansion for $1/(z-1)$ around $z=-1$ is

$$\begin{aligned}
\frac{1}{z-1} &= \frac{1}{(z+1)-2} = \left(\frac{1}{z+1}\right)\left(\frac{1}{1-\frac{2}{z+1}}\right) \\
&= \frac{1}{z+1}\sum_{n=0}^{\infty}\frac{2^n}{(z+1)^n} \\
&= \sum_{n=0}^{\infty}\frac{2^n}{(z+1)^{n+1}} \qquad (4)
\end{aligned}$$

Since, this series converges uniformly in this region, it can be differentiated term-by-term.

$$\begin{aligned}
\frac{1}{(z-1)^2} &= -\frac{d}{dz}\left(\frac{1}{z-1}\right) = -\frac{d}{dz}\sum_{n=0}^{\infty}\frac{2^n}{(z+1)^{n+1}} \\
&= -\sum_{n=0}^{\infty}\frac{(-(n+1))(2^n)}{(z+1)^{n+2}} \\
&= \sum_{n=0}^{\infty}\frac{(n+1)(2^n)}{(z+1)^{n+2}} \qquad (5)
\end{aligned}$$

Using the results of (4) and (5), the Laurent series expansion of the function $f(z) := (z^2-1)^{-1}$ can be readily found:

$$\begin{aligned}
f(z) &= \frac{1}{(z+1)^2}\cdot\frac{1}{(z-1)^2} = \frac{1}{(z+1)^2}\sum_{n=0}^{\infty}\frac{(n+1)(2^n)}{(z+1)^{n+2}} \\
&= \sum_{n=0}^{\infty}\frac{(n+1)(2^{n+1})}{(z+1)^{n+4}}.
\end{aligned}$$

2. Find the principal parts of the Laurent series expansions for the following functions around the indicated points.

   (i) Let $f(z) := (z^2\sin z)^{-1}$ and consider the Laurent series expansion about $a=0$. The function z^2 has a double zero at $z=0$ while the function $\sin z$ has a simple zero wherever $z=n\pi$ $(n\in\mathbb{Z})$. Thus, the function f can be expanded as follows:

$$\begin{aligned}
f(z) &= \frac{1}{z^2}(\sin z)^{-1} = \frac{1}{z^2}\left(z-\frac{z^3}{3!}+O(z^5)\right)^{-1} = \frac{1}{z^3}\left(1-\frac{z^2}{3!}+O(z^4)\right)^{-1} \\
&= \frac{1}{z^3}\left(1-\left(-\frac{z^2}{3!}+O(z^4)\right)\right) = \frac{1}{z^3}+\frac{1}{6z}+O(z).
\end{aligned}$$

   Thus, the principal part of $f(z)$ is

$$\frac{1}{z^3}+\frac{1}{6z}.$$

(iii) Let $f(z) := e^{-1/z^4}$ and consider the Laurent series expansion around $z = 0$. Since the Maclaurin series expnasion for e^z is well-known, we can use it to find the Laurent series expansion for f. The Laurent series is

$$f(z) = \sum_{n=0}^{\infty} \frac{(-z^{-4})^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{z^{4n}}$$

and the function f has an essential isolated singularity at $z = 0$. Hence, the principal part of $f(z)$ expanded around $a = 0$ is

$$f(z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{1}{z^{4n}} \tag{6}$$

(v) $a = ib$, $f(z) := e^{iz}(z^2 + b^2)^{-3}$. The easiest way to approach this problem is to write $e^{iz} = e^{i(z+ib)} \cdot e^{-b}$. The problem can then be broken into two parts, the first depending on the Taylor series expansion of $e^{i(z-ib)}$ and the second depending upon the Laurent series for $(z + ib)^{-3}$ about $a = ib$:

$$\begin{aligned}
e^{i(z-ib)} &= \sum_{n=0}^{\infty} \frac{i^n}{n!}(z-ib)^n. && (7)\\
\frac{1}{(z+ib)^3} &= \frac{1}{2}\frac{d^2}{dz^2}\left(\frac{1}{z+ib}\right) = \frac{1}{2}\frac{d^2}{dz^2}\left(\frac{1}{(z-ib)+2ib}\right) \\
&= \frac{1}{4ib}\frac{d^2}{dz^2}\left(\frac{1}{1+\frac{z-ib}{2ib}}\right) \\
&= \frac{1}{4ib}\frac{d^2}{dz^2}\left(\sum_{n=0}^{\infty}(-1)^n \frac{(z-ib)^n}{(2ib)^n}\right) \\
&= \frac{i}{16b^3}\sum_{n=0}^{\infty} \frac{(-1)^n(n+1)(n+2)(z-ib)^{n-2}}{(2ib)^n}. && (8)
\end{aligned}$$

Combining the results of (7) and (8) gives the principal part of f:

$$\begin{aligned}
f(z) &= e^{-b}(z-ib)^{-3}e^{i(z-ib)}(z+ib)^{-3} \\
&= e^{-b}(z-ib)^{-3}\left(1 + i(z-ib) - \frac{1}{2}(z-ib)^2 + O((z-ib)^3)\right) \\
&\quad \cdot \left(\frac{i}{8b^3} - \frac{3}{16b^4}(z-ib) - \frac{3i}{16b^5}(z-ib)^2 + O((z-ib)^3)\right) \\
&= \frac{e^{-b}i/(8b^3)}{(z-ib)^3} - \frac{e^{-b}(2b+3)/(16b^4)}{(z-ib)^2} \\
&\quad - \frac{e^{-b}(b^2+3b+1)i/(16b^5)}{(z-ib)} + h(z)
\end{aligned}$$

where $h(z)$ has a removable singularity and is holomorphic at $z = ib$. Thus, the principal part of f at $a = ib$ is

$$\frac{e^{-b}i}{8b^3}(z-ib)^{-3} - \frac{e^{-b}(2b+3)}{16b^4}(z-ib)^{-2} - \frac{e^{-b}(b^2+3b+1)i}{16b^5}(z-ib)^{-1}.$$

5. Classify the singularities of these functions:

   (i) $f(z) := z^{-3}(z^2+1)^{-1} = z^{-3}(z+i)^{-1}(z-i)^{-1}$. This function has a triple pole at $z = 0$ and simple poles at $z = i$ and $z = -i$.

   (iii) $f(z) := e^{iz}(z^2+z+1)^{-2}$. This function can be written in the form $f(z) = e^{iz}(z-\alpha)^{-2}(z-\overline{\alpha})^{-2}$, where $\alpha := \frac{1}{2}(-1+\sqrt{3})$. Clearly, since e^{iz} is entire and everywhere nonzero, f is holomorphic everywhere except at the double poles $z = \alpha$ and $z = \overline{\alpha}$.

   (v) $f(z) := z \sin z(\cos z - 1)^{-1}$. The numerator $z \sin z$ is entire, has a zero of order 2 at $z = 0$ and has zeroes of order 1 at the points $z = k\pi$ ($k \in \mathbb{Z}\backslash\{0\}$). The denominator $(\cos z - 1)$ has double zeroes at $z = 2n\pi$ ($n \in \mathbb{Z}$). Thus, f has simple poles at the points $z = 2n\pi$ ($n \in \mathbb{Z}\backslash\{0\}$) and a removable singularity at $z = 0$.

7. (i) The function $\cos(\pi z/2)$ is entire has simple zeroes at the points $z = 2k+1$ ($k \in \mathbb{Z}$). The function $\sin z/z$ has a removable singularity at $z = 0$ and has zeroes at $z = n\pi$ ($n \in \mathbb{Z}$). Defining

$$f_1(z) := \frac{\sin z}{z \cos^4(\pi z/2)},$$

   it is clear that f_1 has poles of order 4 at $z = 2k+1$ ($k \in \mathbb{Z}$) and has a removable singularity at $z = 0$.

   (ii) A function that is holomorphic everywhere except at the double poles at ± 1, $\pm i$ is $(z-1)^{-1}(z+1)^{-1}(z-i)^{-1}(z+i)^{-1} \equiv (z^4-1)^{-1}$. To include a double pole at ∞, multiply by z^2. Thus,

$$f_2(z) := \frac{z^2}{z^4-1}$$

   is the desired function.

   (iii) The functions $e^{1/z}$, $e^{1/(z-1)}$ and e^z have isolated essential singularities at the points $z = 0$, $z = 1$ and $z = \infty$ respectively in the extended complex plane. Thus, the function

$$f_3(z) := e^{z+\frac{1}{z}+\frac{1}{z-1}} = \exp\left(\frac{z^3-z^2+2z-1}{z(z-1)}\right)$$

   is holomorphic except at the isolated essential singularities $z = 0, 1, \infty$.

(iv) The function $\sin(1/(z-1))$ has a non-isolated essential singularity at $z = 1$ and has simple zeroes at the points $z = 1 + 1/(k\pi)$ $(k \in \mathbb{Z})$. The function $\sin(1/(z+1))$ has a non-isolated essential singularity at $z = -1$ and has simple zeroes at the points $z = -1 + 1/(k\pi)$ $(k \in \mathbb{Z})$. (Notice that none of the zeroes of $\sin(1/(z-1))$ coincide with the zeroes of $\sin(1/(z+1))$.) Thus, the function

$$f_4(z) := \frac{1}{\sin^2((z-1)^{-1})\sin^2((z+1)^{-1})}$$

has non-isolated essential singularities at $z = \pm 1$, double poles at the points $1/(k\pi) \pm 1$ $(k \in \mathbb{Z})$.

10. Suppose that f is holomorphic in a punctured disk centred at $a \in \mathbb{C}$. Fix $w \in \mathbb{C}$ and assume

$$(\exists \epsilon, r > 0) \ni [\forall z \in D'(a; r), |f(z) - w| \geq \epsilon]. \tag{9}$$

Without loss of generality, the function $g(z) := (f(z) - w)^{-1}$ is holomorphic in $D'(a; r)$; as such, g has a Laurent series expansion in $D'(a; r)$ with coefficients c_n as given in Laurent's theorem. For any integer $n > 0$ and $0 < s < r$,

$$\begin{aligned} |c_{-n}| &= \left| \frac{1}{2\pi i} \int_{\gamma(0;s)} \frac{g(z)dz}{(z-a)^{-n+1}} \right| \\ &\leq \frac{2\pi s}{2\pi \epsilon s^{-n+1}} = s^n, \end{aligned}$$

using the Estimation theorem and the hypothesis (9). Since $n > 0$ and s can be made arbitrarily small, it follows that $c_{-n} = 0$. Thus, the principal part of the Laurent series expansion for g is zero. The function g has a removable singularity at $z = a$ and is holomorphic in $D(a; r)$.

Since $f(z) = w + 1/g(z)$, there are two possibilities:

- $g(a) \neq 0$, in which case f is holomorphic at a.
- g has a zero of order k at m in which case f has a pole of order m at a. (It is not possible for g to have a zero of infinite order at a because, by Taylor's theorem, g would vanish in an open neighbourhood of a, contradicting (9).)

In either of these cases, f does not have an essential singularity at $z = a$.

The claim just established implies the Casorati-Weierstrass theorem. The requirement (9) is equivalent to stating that there is a $w \in \mathbb{C}$ and a punctured neighbourhood $D'(a; r)$ such that w is not a limit point of $f(D'(a; r))$ and we proved that this means that a cannot be an essential singularity of f. This means that, if a is an essential singularity of f, for every $w \in \mathbb{C}$, there is a sequence $< a_n >$ converging to a such that $< f(a_n) >$ converges to w.