Math 322: Assignment 6 Key

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Priestly, Chapter 7

1. The residues below are calculated using the formulas on p. 110-111.

	$f(z)$	poles	residues
i	$z^{-3}(z^2+1)^{-1}$	$0, i, -i$	$-1, 1/2, 1/2$
ii	$z^{-2}(1-e^{iz})$	0	$-i$
iii	$(z^2+z+1)^{-2}e^{iz}$	$\frac{1}{2}(-1 \pm i\sqrt{3})$	$-\frac{i}{9}e^{(i(-1\pm i\sqrt{3})/2)}(3 \pm 2\sqrt{3})$
iv	$(z-1)^{-1}\cot(\pi z)$	$1, n\ (n \in \mathbb{Z} \setminus \{1\})$	$0, ((n-1)\pi)^{-1}$
v	$(z \sin z)(\cos z - 1)^{-1}$	$2n\pi\ (n \in \mathbb{Z} \setminus \{0\})$	$-2n\pi$
vi	$(1-e^{z^2})^{-1}$	$0, \pm\sqrt{n\pi}(1+i)\ (n \in \mathbb{N})$	$0, \pm\frac{1}{4\sqrt{n\pi}}(-1+i)$

2. (i) Let $f(z) := (z-1)^{-2}(z^2+1)^{-1} = (z-1)^{-2}(z-i)^{-1}(z+i)^{-1}$. Then, f has simple poles at $z = \pm i$ and a double pole at $z = 1$. The residues are calculated as follows:

$$\begin{aligned} \text{res}\{f(z); i\} &= \frac{1}{(i-1)^2(i+i)} = \frac{1}{4} \\ \text{res}\{f(z); -i\} &= \frac{1}{(-i-1)^2(-i-i)} = \frac{1}{4} \\ \text{res}\{f(z); 1\} &= \frac{1}{-(1^2+1)(2-1)} = -\frac{1}{2} \end{aligned}$$

Since the sum of these residues is zero, it follows that

$$\int_{\gamma(0;2)} f(z) = 0,$$

by Cauchy's residue theorem (since all the poles lie inside the curve).

(ii) Let $f(z) := (1+e^z)^{-1}$. The function f has simple poles at the points $z = (2n+1)\pi i$ where $n \in \mathbb{Z}$. Only two of these poles lie within the contour $\gamma(0;8)$, namely $z = \pm\pi i$. These are covert poles of type II which can be expressed as $f(z) = g(z)/h(z)$, where $g(z) = 1$ and $h(z) = 1 + e^z$. The residue at any of the poles is

$$\text{res}\{f(z); (2n+1)\pi i\} = \frac{g((2n+1)\pi i)}{h'((2n+1)\pi i)} = e^{-(2n+1)\pi i} = -1.$$

Thus, by Cauchy's residue theorem,

$$\int_{\gamma(0;8)} \frac{dz}{1+e^z} = 2\pi i((-1)+(-1)) = -4\pi i.$$

3. Let $f(z) := z(2z^4+5z^2+2)^{-1} = z(2z^2+1)^{-1}(z^2+2)^{-1}$. The function f has four simple poles at the complex numbers $z = \pm\sqrt{2}i$ and $z = \pm(1/\sqrt{2})i$. Of these, only two lie within the contour $\gamma(0;1)$, namely $z = i/\sqrt{2}$ and $z = -i/\sqrt{2}$. Taking $p(z) := z/(z^2+2)$ and $q(z) := (2z^2+1)$, we can think of both of these poles as covert simple poles to calculate the corresponding residues:

$$\begin{aligned}
\mathrm{res}\{f(z); i/\sqrt{2}\} &= \frac{p(i/\sqrt{2})}{q'(i/\sqrt{2})} = \frac{i/\sqrt{2}}{-1/2+2}\frac{1}{4i/\sqrt{2}} = \frac{1}{6} &(1)\\
\mathrm{res}\{f(z); -i/\sqrt{2}\} &= \frac{p(-i/\sqrt{2})}{q'(-i/\sqrt{2})} = \frac{-i/\sqrt{2}}{-1/2+2}\frac{1}{-4i/\sqrt{2}} = \frac{1}{6}. &(2)
\end{aligned}$$

Applying Cauchy's residue theorem with the aid of (1) gives

$$\int_{\gamma(0;1)} f(z)dz = \frac{2\pi}{3}i. \tag{3}$$

the integral can also be computed explicitly:

$$\begin{aligned}
\int_{\gamma(0;1)} f(z)dz &= \int_0^{2\pi} \frac{e^{i\theta}\cdot ie^{i\theta}d\theta}{2e^{4i\theta}+5e^{2i\theta}+2} = i\int_0^{2\pi} \frac{d\theta}{2(e^{2i\theta}+e^{-2i\theta})+5}\\
&= i\int_0^{2\pi} \frac{d\theta}{4\cos 2\theta + 5} = i\int_0^{2\pi} \frac{d\theta}{1+8\cos^2\theta}. &(4)
\end{aligned}$$

Equating the right-hand sides of (3) and (4) gives

$$\int_0^{2\pi} \frac{d\theta}{1+8\cos^2\theta} = \frac{2\pi}{3}.$$

4. Let I denote the real-valued integral definite $\int_0^{2\pi}(\cos^4\theta+\sin^4\theta)^{-1}d\theta$. On the contour $\gamma(0;1)$, $z = e^{i\theta}$, so $\cos\theta = (e^{i\theta}+e^{-i\theta})/2 = (z+z^{-1})/2$ and $\sin\theta = (e^{i\theta}-e^{-i\theta})/2 = (z-z^{-1})/2$. Further, $d\theta = -(i\,dz)/z$. Thus,

$$\begin{aligned}
I &= \int_0^{2\pi} (\cos^4\theta + \sin^4\theta)d\theta\\
&= \int_{\gamma(0;1)} \left(\left(\frac{z+z^{-1}}{2}\right)^2 + \left(\frac{z-z^{-1}}{2}\right)^2\right)\left(\frac{-idz}{z}\right)\\
&= -\frac{i}{8}\int_{\gamma(0;1)} z^{-1}(z^4+6+z^{-4})dz\\
&= -\frac{i}{8}\left(\int_{\gamma(0;1)} z^3dz + \int_{\gamma(0;1)} 6z^{-1}dz + \int_{\gamma(0;1)} z^{-4}dz\right)\\
&= -\frac{i}{8}\left(0+6(2\pi i)+0\right) = \frac{3\pi}{2},
\end{aligned}$$

where the result from Example 3.7 (p. 34) can be used rather than computing residues.

5. Estimate the following limits.

   (i) Let $f(z) := e^{iz}(z^4+z^3+z^2+z+1)^{-1}$ and, for $R>0$, let Γ_R denote the semi-circle described by Re^{it} $(t \in [0,\pi])$ in the upper half of the complex plane. Provided R is sufficiently large that $R^4 - R^3 - R^2 - R - 1 > 0$, it follows that

$$\begin{aligned}
\left|\int_{\Gamma_R} f(z)dz\right| &= \left|\int_0^{\pi} \frac{e^{i(R\cos t + iR\sin t)}(iRe^{it})dt}{(Re^{it})^4+(Re^{it})^3+(Re^{it})^2+(Re^{it})+1}\right| \\
&\leq \int_0^{\pi} \frac{e^{-R\sin t}Rdt}{|R^4-R^3-R^2-R-1|} \leq \frac{(1)R}{R^4-R^3-R^2-R-1}(\pi) \\
&= \frac{\pi R}{R^4-R^3-R^2-R-1}.
\end{aligned}$$

   Thus, as $R \to \infty$, the term on the right goes to zero, so the limit of the integral vanishes also.

   (ii) Let $p(z)$ be a polynomial with complex coefficients of degree m and let $q(z)$ be a poynomial with complex coefficients of degree n, where $n > m+1$. Then, if $|z| = R$ is sufficiently large, we can write

$$|p(z)| = \left|a_m z^m + \sum_{j=0}^{m-1} a_j z^j\right| \leq |a_m|R^m + \sum_{j=0}^{m-1} |a_j|R^j \leq KR^m,$$

   where a_j are the coeeficients of $p(z)$. Similarly, if $b_0, b_1, \ldots, b_n$ are the coefficients of $q(z)$,

$$|q(z)| \geq \left||b_n|R^n - \sum_{j=0}^{n-1}|b_j|R^j\right|$$

   Thus,

$$\begin{aligned}
\left|\int_{\gamma(0;R)} \frac{p(z)}{q(z)}dz\right| &\leq \int_0^{2\pi} \frac{(KR^m)(Rd\theta)}{\left||b_n|R^n - \sum_{j-0}^{n-1}|b_j|R^j\right|} \\
&\leq \frac{2\pi K}{\left||b_n|R^{n-m-1} - \sum_{j-0}^{n-1}|b_j|R^{j-m-1}\right|}.
\end{aligned}$$

   The right-hand side tends to zero as $R \to \infty$ since $n-m-1>0$ by assumption. Thus, the integral vanishes.

(iii) Let $f(z) := (z-1)e^{iz}(z^2-1)^{-1}$ and, for $R > 0$, let Γ_R denote the semi-circle described by Re^{it} $(t \in [0,\pi])$. Then, for $R > 1$,

$$\begin{aligned}
\left|\int_{\Gamma_R} f(z)dz\right| &= \left|\int_0^\pi \frac{(Re^{it}-1)e^{i(R\cos t + iR\sin t)}(iRe^{it})dt}{(Re^{it})^2+1}\right| \\
&\leq \int_0^\pi \frac{R(R-1)e^{-R\sin t}dt}{|R^2-1|} \leq \frac{R}{R+1}\int_0^\pi e^{-R\sin t}dt \\
&= \frac{2R}{R+1}\int_0^{\pi/2} e^{-R\sin t}dt \leq \frac{2R}{R+1}\int_0^{\pi/2} e^{-2Rt/\pi}dt \\
&= \frac{\pi(1-e^{-R})}{R-1}.
\end{aligned}$$

Jordan's inequality is needed in the 3rd line because the upper bound is $O(R/R)$ without it. Clearly, the term on the right goes to zero as $R \to \infty$ so the limit of the integral vanishes.

(iv) Let $f(z) := \csc(\pi z)$ and consider a semi-circle Γ_ε in the uppper half complex plane described by εe^{it} $(t \in [0,\pi])$. Then, $f(z)$ has a simple pole at $z = 0$ and the residue is $\operatorname{res}\{f;0\} = 1/\pi$ since the Laurent series expansion is

$$f(z) = \csc(\pi z) = (\sin(\pi z))^{-1} = \left(\frac{1}{\pi}z^{-1} + \frac{\pi}{6}z + O(z^3)\right).$$

It follows from the Lemma on p. 115 (indentation at a simple pole) that, since $f \in H(D'(0;1))$ and $\theta_1 = 0$, $\theta_2 = \pi$ for this case,

$$\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} \csc(\pi z)dz = i(1/\pi)(\pi - 0) = i.$$

6. Let f be a function that is holomorphic everywhere except at the double poles $z = 1$ and $z = -1$. Suppose that the residues are $a := \operatorname{res}\{f(z);+1\}$ and $b := \operatorname{res}\{f(z);-1\}$. Further, suppose that there exists $K > 0$ such that $|z^2 f(z)| \leq K$ for $|z|$ sufficiently large.

Thus, taking a contour $\gamma(0;R)$ with $R > 1$ sufficiently large,

$$\begin{aligned}
2\pi|a+b| &= \left|\int_{\gamma(0;R)} f(z)dz\right| \leq \int_0^{2\pi} |f(Re^{it})|Rdt \\
&\leq \frac{2\pi K}{R}.
\end{aligned}$$

Since the right-hand side goesto 0 as $R \to \infty$, it follows that $a + b = 0$.

The function

$$g(z) := f(z) - \frac{1}{(z-1)} - \frac{\alpha}{(z-1)^2} + \frac{1}{(z+1)^2} - \frac{\beta}{(z+1)^2}$$

has removable singularities and is entire. Further, since $f(z)$ is bounded, it follows that $g(z) \to 0$ as $|z| \to \infty$. Taking R sufficiently large, it is clear, then, that g is bounded on the compact set $D(0;R)$ and tends to zero outside. Since g is bounded and entire, Liouiville's theorem asserts that g is a constant-valued function. $g(z) \to 0$ as $|z| \to \infty$, so $g(z) = 0$. As a consequence, if $a = 1$ it follows that $b = -1$ so

$$f(z) = \frac{1}{(z-1)} + \frac{\alpha}{(z-1)^2} - \frac{1}{(z+1)^2} + \frac{\beta}{(z+1)^2}$$

for some complex constants a, b, α and β. If $a = 1$ then, from above, $b = -1$. Using the conditions $f(2i) = f(-2i) = 0$, we can determine the constants α and β uniquely. The linear system obtained for α and β has as its solution $\alpha = \beta = -5/3$. Hence,

$$f(z) = \frac{1}{(z-1)} - \frac{5}{3(z-1)^2} - \frac{1}{(z+1)^2} - \frac{5}{3(z+1)^2}.$$

7. let $p(w)$ and $q(w)$ be polynomials with complex coefficients of degrees m and n respectively. Assume $n \geq m+1$. Further, assume that the polynomials $p(w)$ and $q(w)$ have distinct zeroes (otherwise, the common zeroes could be divided out) and that $q(w)$ has n aimple zeroes. Taking $\gamma(0;R)$ sufficiently large such that all the zeroes $b_1, b_2, ...b_n$ of $q(w)$ lie within $D(0;R)$. Fixing $z \in D(0;R)\backslash\{b_1, b_2, ...b_n\}$, it follows that the polynomial

$$f(w) := \frac{p(w)}{q(w)(z-w)}$$

has $n+1$ simple zeroes in $D(0;R)$ namely $b_1, b_2, ...b_n$ and z. Thus, since each of these poles is a simple pole of covert type, the methods on p. 110 for evaluating residues tell us that

$$\text{res}\{f; b_k\} = \frac{p(b_k)}{q'(b_k)}(z-b_k)^{-1} \text{ and } \text{res}\{f; z\} = -\frac{p(z)}{q(z)}.$$

The degree of the denominator of $f(w)$ is $N := n+1$ and the degree of the numerator of $f(w)$ is $m < n+2 = N+1$. From the result of problem 7.5(ii) and the Cauchy residue theorem, it follows that

$$\int_{\gamma(0;R)} f(w)dw = 2\pi i \left(\sum_{k=1}^{n} \frac{p(b_k)}{q'(b_k)}(z-b_k)^{-1} - \frac{p(z)}{q(z)} \right) \to 0$$

as $R \to \infty$. Since the term in the brackets does not depend on R,

$$\frac{p(z)}{q(z)} = \sum_{k=1}^{n} \frac{p(b_k)}{q'(b_k)}(z-b_k)^{-1}$$

is the partial fraction decomposition of $p(z)/q(z)$. Thus, taking $p(z) := 1 - z^2$ and $q(z) := 1 + z^4$, $q'(z) = 4z^3$. The distinct simple zeroes of $q(z)$ are $b_1 = \omega = e^{\pi i/4}, b_2 = \omega^3, b_3 = \omega^5$ and $b_4 = \omega^7$ where $\omega := e^{\pi i/4}$ is one of the fourth roots of -1. Using the above result,

$$\begin{aligned}
\frac{p(z)}{q(z)} &= \frac{1-z^2}{1+z^4} = \sum_{k=1}^{n} \frac{p(b_k)}{q'(b_k)}(z-b_k)^{-1} \\
&= \frac{1-(\omega)^2}{4(\omega)^3}(z-\omega)^{-1} + \frac{1-(\omega^3)^2}{4(\omega^3)^3}(z-\omega^3)^{-1} \\
&\quad + \frac{1-(\omega^5)^2}{4(\omega^5)^3}(z-(\omega^5))^{-1} + \frac{1-(\omega^7)^2}{4(\omega^7)^3}(z-(\omega^7))^{-1} \\
&= -\frac{\sqrt{2}}{4(z-\omega)} + \frac{\sqrt{2}}{4(z-\omega^3)} + \frac{\sqrt{2}}{4(z-\omega^5)} - \frac{\sqrt{2}}{4(z-\omega^7)}.
\end{aligned}$$

12. For $n = 3, 4, \ldots$, define $r := 1 + (2/(n-1))^{1/2} > 1$. By the binomial theorem,

$$\begin{aligned}
r^n - 1 &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{2}{n-1}\right)^{k/2} - 1 \\
&= \left(1 + n\left(\frac{2}{n-1}\right)^{\frac{1}{2}} + \frac{n(n-1)}{2}\left(\frac{2}{n-1}\right)^{\frac{2}{2}} \right. \\
&\quad \left. + \sum_{k=3}^{n} \binom{n}{k} \left(\frac{2}{n-1}\right)^{k/2}\right) - 1 \\
&> n\left(1 + \left(\frac{2}{n-1}\right)^{\frac{1}{2}}\right) = nr. \qquad (5)
\end{aligned}$$

Defining $f(z) := z^n - 1$ and $g(z) := nz$, for $z \in \gamma^*(0; r)$,

$$|f(z)| = |z^n - 1| \geq |z|^n - 1 = r^n - 1 > nr = n|z| = |g(z)| \qquad (6)$$

using the equation (5). Further, since $r > 1$, the function f has exactly n distinct simple zeroes within $\gamma(0; r)$, namely the nth roots of unity. The inequality (6) together with Rouché's theorem implies that the function $f + g$ has the same number of zeroes as f within $\gamma(0; r)$. Thus, the function $(f + g)(z) := z^n + nz - 1$ has exactly n zeroes within the disk $\gamma(0; r)$.