22. $\sum_{n=1}^{\infty}\left[\frac{1}{2^{n-1}}+\frac{2}{3^{n-1}}\right]=\sum_{n=0}^{\infty}\left[\frac{1}{2^{n}}+\frac{2}{3^{n}}\right]=\sum_{n=0}^{\infty}\frac{1}{2^{n}}+2\sum_{n=0}^{\infty}\frac{1}{3^{n}}$ is the sum of two convergent geometric series and converges to $\frac{1}{1-\frac{1}{2}}+\frac{2}{1-\frac{1}{3}}=5$.

28. $\sum_{n=1}^{\infty}\ln\left(\frac{n}{2n+5}\right)$ diverges because $\lim_{n\to\infty}\ln\left(\frac{n}{2n+5}\right)=\ln\frac{1}{2}\neq 0$.

34. $\frac{1}{n(n+1)(n+2)}=\frac{1/2}{n}-\frac{1}{n+1}+\frac{1/2}{n+2}=\frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+1}\right)-\frac{1}{2}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$.

The $n^{\underline{th}}$ partial sum is $s_n=\sum_{i=1}^{n}\frac{1}{i(i+1)(i+2)}=\sum_{i=1}^{n}\frac{1}{2}\left(\frac{1}{i}-\frac{1}{i+1}\right)-\sum_{i=1}^{n}\frac{1}{2}\left(\frac{1}{i+1}-\frac{1}{i+2}\right)=$

$=\frac{1}{2}\left(1-\frac{1}{n+1}\right)-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{n+2}\right)$ since the sums telescope.

Since $\lim_{n\to\infty}s_n=\frac{1}{4}$, the series converges to $\frac{1}{4}$.

44. $\sum_{n=0}^{\infty}3^n x^n=\sum_{n=0}^{\infty}(3x)^n$ converges if and only if $|3x|<1$.

This condition is equivalent to $|x|<\frac{1}{3}$, or to $-\frac{1}{3}<x<\frac{1}{3}$.

For such values of x the series converges to $\frac{1}{1-3x}$.

48. $\sum_{n=0}^{\infty}\tan^n x=\sum_{n=0}^{\infty}(\tan x)^n$ converges if and only if $|\tan x|<1$.

This condition is equivalent to $\left(k-\frac{1}{4}\right)\pi<x<\left(k+\frac{1}{4}\right)\pi$ (for some integer k).

For such values of x the series converges to $\frac{1}{1-\tan x}$.

64. $\sum(a_n+b_n)$ **might** converge if $\sum a_n$ and $\sum b_n$ both diverged.
For example consider what happens if $\sum a_n$ diverges and $b_n=-a_n$.

Then again it might **not** converge.
Consider what happens if $\sum a_n$ diverges and $b_n=a_n$.

54. $a_n = \frac{3n+4}{2n+5}$. Notice $a_n - \frac{3}{2} = \frac{3n+4}{2n+5} - \frac{3}{2} = -\frac{7}{2(2n+5)}$.
Then $\left\{a_n - \frac{3}{2}\right\}$ is clearly increasing, because of the minus sign, and so is $\{a_n\}$ itself.

56. If $a_n = 3 + \frac{(-1)^n}{n}$ then $\{a_n\}$ is not monotonic. When n is odd, $a_n < 3$, and when n is even, $a_n > 3$. So the terms oscillate from one side of 3 to the other, forever.

66. $|r^n - 0| = |r^n|$. Assume $r \neq 0$. Then if ε is any positive number, we can make $|r^n| < \varepsilon$ by making $n\ln|r| < \ln\varepsilon$. For $0 < |r| < 1$, $\ln|r| < 0$, so this inequality $n\ln|r| < \ln\varepsilon$ is equivalent to $n > \frac{\ln\varepsilon}{\ln|r|}$. Take any integer N for which $N \geq \frac{\ln\varepsilon}{\ln|r|}$.
If $n > N$ then $n > \frac{\ln\varepsilon}{\ln|r|}$, so $n\ln|r| < \ln\varepsilon$, $\ln|r^n| < \ln\varepsilon$, and $|r^n - 0| = |r^n| < \varepsilon$.

What if $r = 0$? The above argument won't work since $\ln|r|$ doesn't exist. But to get $|0^n - 0| < \varepsilon$ is easy; it holds for **every** positive integer n. So take $N = 0$ for instance.

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8. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots = \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n$ is a geometric series.
Since $\left|-\frac{1}{2}\right| < 1$, it converges and its sum is $\frac{1}{1-(-1/2)} = \frac{2}{3}$.

10. $-\frac{81}{100} + \frac{9}{10} - 1 + \frac{10}{9} - \cdots = \sum_{n=0}^{\infty}\left(-\frac{81}{100}\right)\left(-\frac{10}{9}\right)^n$ is a geometric series.
Since $\left|-\frac{10}{9}\right| \geq 1$, it diverges.

14. $\sum_{n=1}^{\infty}\frac{1}{e^{2n}} = \sum_{n=1}^{\infty}\left(\frac{1}{e^2}\right)^n$ is a geometric series.
Since $\left|\frac{1}{e^2}\right| < 1$, it converges and its sum is $\frac{\frac{1}{e^2}}{1-\frac{1}{e^2}} = \frac{1}{e^2-1}$.

18. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3^{2n}}{2^{3n+1}} = \sum_{n=1}^{\infty}\left(-\frac{1}{2}\right)\left(-\frac{9}{8}\right)^n$ is a geometric series.
Since $\left|-\frac{9}{8}\right| \geq 1$, it diverges.

This will make $-\varepsilon < \arctan(2n) - \frac{\pi}{2}$. On the other hand, $\arctan(2n) - \frac{\pi}{2} < 0 < \varepsilon$ regardless of how large or small n may be. So if $n > N \ge \frac{1}{2}\tan\left(\frac{\pi}{2} - \varepsilon\right)$ then $-\varepsilon < \arctan(2n) - \frac{\pi}{2} < \varepsilon$, and $\left|\arctan(2n) - \frac{\pi}{2}\right| < \varepsilon$.

26. If $a_n = \frac{n!}{(n+2)!}$ then $\{a_n\}$ converges to 0.

To see why, notice that $a_n = \frac{1}{(n+1)(n+2)}$ and thus $0 < a_n < \frac{1}{n^2}$.

To make $|a_n - 0| = |a_n| < \varepsilon$, where ε is any positive number, it will be more than sufficient to have $n > N$, where N is any integer for which $N \ge \frac{1}{\sqrt{\varepsilon}}$. Then if $n > N$, we will have $0 < a_n < \frac{1}{n^2} < \frac{1}{N^2} \le \varepsilon$, so that $|a_n - 0| = |a_n| < \varepsilon$.

Actually this is overkill. By solving a quadratic equation you can show that $N \ge -1.5 + \sqrt{0.25 + \varepsilon^{-1}}$ will work, and that this is the best one can do.

30. If $a_n = \frac{\ln(2 + e^n)}{3n}$ then $\{a_n\}$ converges to $\frac{1}{3}$.

Notice $\left|\frac{\ln(2+e^n)}{3n} - \frac{1}{3}\right| = \frac{\ln(2+e^n) - n}{3n} = \frac{\ln(2+e^n) - \ln e^n}{3n} = \frac{1}{3n}\ln\frac{2+e^n}{e^n} = \frac{1}{3n}\ln(1+2e^{-n})$.

The continuity of the natural logarithm function insures that if $(1 + 2e^{-n})$ gets close enough to 1, $\ln(1 + 2e^{-n})$ will get as close as we want to 0. The $(3n)$ in the denominator makes things even better. And we can bring $(1 + 2e^{-n})$ as close as we want to 1, since e^n can be made as large as we please by taking n large enough.

36. If $a_n = \frac{n\cos n}{n^2 + 1}$ then $\{a_n\}$ converges to 0.

Since $|\cos n| \le 1$, $|a_n - 0| = |a_n| \le \frac{n}{n^2+1}$. We can make $\frac{n}{n^2+1} < \varepsilon$, where ε is any positive number, just by taking $n > N$ where N is any integer for which $N \ge \frac{1}{\varepsilon}$.

If $n > N$ then $n > \frac{1}{\varepsilon}$ so $\frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n} < \varepsilon$.

(This argument shows that probably we can get away with a slightly smaller value of N than $\frac{1}{\varepsilon}$, since we have “thrown away” the $+1$ in $n^2 + 1$. Sometimes $|\cos n|$ is a lot smaller than 1, too, but then sometimes it is quite close to 1 so that won’t help.)

48. If $a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!}$ then $\{a_n\}$ diverges to $+\infty$.

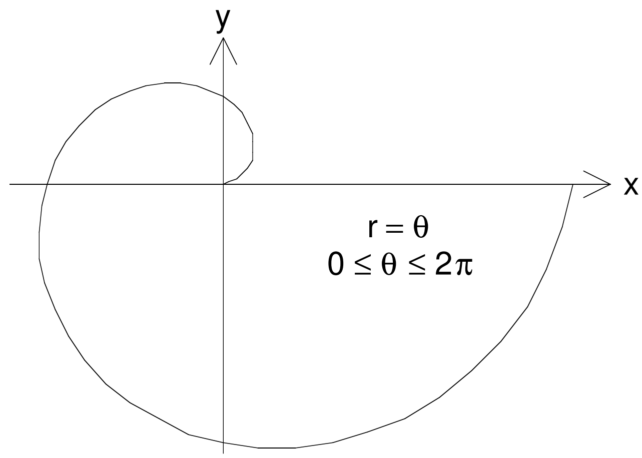
To see why, write $a_n = \frac{1}{1} \cdot \frac{3}{2} \cdot \frac{5}{3} \cdots \frac{2n-1}{n}$.

Notice $\frac{3}{2} < \frac{5}{3} < \frac{7}{4} < \cdots < \frac{2n-1}{n}$, so $a_n > \left(\frac{3}{2}\right)^{n-1} = 1.5^{n-1}$, when $n > 2$.

Since 1.5^{n-1} gets as large as you please and a_n is larger, $\{a_n\}$ diverges to $+\infty$, almost as fast as 2^{n-1} (not quite that fast, since $\frac{3}{2} < 2$, $\frac{5}{3} < 2$, $\cdots$, and $\frac{2n-1}{n} < 2$).

A good way to illustrate this with a graph would be to look at $\frac{1}{a_n}$ instead of a_n.

46. If $r = \theta$ then $\frac{dr}{d\theta} = 1$.
The length of the portion of the graph of $r = \theta$ with $0 \le \theta \le 2\pi$ is
$L = \int_0^{2\pi} \sqrt{\theta^2 + 1}\, d\theta =$
$= \frac{1}{2}\left(\theta\sqrt{\theta^2+1} + \ln\left(\theta + \sqrt{\theta^2+1}\right)\right)\Big]_0^{2\pi} =$
$= \pi\sqrt{4\pi^2+1} + \frac{1}{2}\ln\left(2\pi + \sqrt{4\pi^2+1}\right)$.
See graph to the right.



For Exercise 46

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2. If $a_n = \left(-\frac{2}{3}\right)^n$ then $a_1 = -\frac{2}{3}$, $a_2 = \frac{4}{9}$, $a_3 = -\frac{8}{27}$, $a_4 = \frac{16}{81}$, and $a_5 = -\frac{32}{243}$.

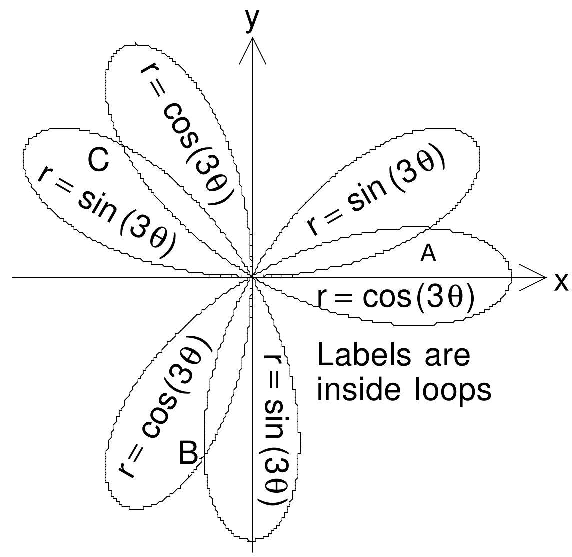
8. For the sequence $\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots\right\}$ it could be that $a_n = \frac{1}{2n}$. There are of course infinitely many other sequences that start out that way for the first four terms.

14. If $a_n = 4\sqrt{n}$ then $\{a_n\}$ diverges to $+\infty$.
To insure that $4\sqrt{n} > M$, where M is any positive number, all we need do is have $n > N$, where N is any integer for which $N \ge \frac{M^2}{16}$. Then if $n > N$, we will have $n > \frac{M^2}{16}$, so $16n > M^2$ and $4\sqrt{n} > M$.

16. If $a_n = \frac{4n-3}{3n+4}$ then $\{a_n\}$ converges to $\frac{4}{3}$.
To see why, notice that $\left|\frac{4n-3}{3n+4} - \frac{4}{3}\right| = \frac{25}{3(3n+4)}$.
To make $\frac{25}{3(3n+4)} < \varepsilon$, where ε is any positive number, all we need do is have $n > N$, where N is any integer for which $N \ge \frac{25-12\varepsilon}{9\varepsilon}$. Then if $n > N$, we will have $n > \frac{25-12\varepsilon}{9\varepsilon}$, so $9n\varepsilon > 25 - 12\varepsilon$, $3\varepsilon(3n+4) > 25$, and $\frac{25}{3(3n+4)} < \varepsilon$.

24. If $a_n = \{\arctan(2n)\}$ then $\{a_n\}$ converges to $\frac{\pi}{2}$.
To make $\left|\arctan(2n) - \frac{\pi}{2}\right| < \varepsilon$, where ε is any positive number, all we need do is have $n > N$, where N is any integer for which $N \ge \frac{1}{2}\tan\left(\frac{\pi}{2} - \varepsilon\right)$. Then if $n > N$, we will have $n > \frac{1}{2}\tan\left(\frac{\pi}{2} - \varepsilon\right)$, so $2n > \tan\left(\frac{\pi}{2} - \varepsilon\right)$, and $\arctan(2n) > \frac{\pi}{2} - \varepsilon$.
(The last step uses the fact that the arctangent function is strictly increasing.)

38. The curve $r = \cos(3\theta)$ has three loops during $0 \le \theta \le 2\pi$, and so does the curve $r = \sin(3\theta)$.



For Exercise 38

If these two curves intersect with $r > 0$ for both curves or with $r < 0$ for both curves then we have $\cos(3\theta) = \sin(3\theta)$, so $\tan(3\theta) = 1$, and θ must be one of $\pi/12$, $5\pi/12$, $9\pi/12 = 3\pi/4$, $13\pi/12$, $17\pi/12$, or $21\pi/12 = 7\pi/4$.

But the points defined by the last three values of θ are **duplicates** of the ones defined by the first three values of θ since for example $\pi/12$ and $13\pi/12$ differ by π (one-half turn), $3\cdot(\pi/12)$ and $3\cdot(13\pi/12)$ differ by 3π (one and one-half turns), and thus the sine values for $3\cdot(\pi/12)$ and for $3\cdot(13\pi/12)$ are negatives of each other, as are the cosine values.

If the graphs of the two curves intersect with the value for r for one graph positive and the value for r for the other graph negative then $\cos(3\theta) = -\sin(3(\theta+\pi))$, but this equation is equivalent to $\cos(3\theta) = \sin(3\theta)$ and therefore leads to no new intersection points. Of course there are lots of intersections at the origin where the values of θ for the two curves need not have any relationship to each other. Since

$\cos(\pi/12) = \sin(5\pi/12) = \frac{\sqrt{3}+1}{2\sqrt{2}}$ (write $\pi/12 = \pi/3 - \pi/4$ to see why),

$\sin(\pi/12) = \cos(5\pi/12) = \frac{\sqrt{3}-1}{2\sqrt{2}}$, and $\cos(3\pi/4) = -\sin(3\pi/4) = -\frac{1}{\sqrt{2}}$,

the rectangular coordinates for the intersection points other than the origin are

$(\cos(\pi/12)/\sqrt{2}, \sin(\pi/12)/\sqrt{2}) = ((\sqrt{3}+1)/4, (\sqrt{3}-1)/4) = A$,

$(\cos(5\pi/12)/(-\sqrt{2}), \sin(5\pi/12)/(-\sqrt{2})) = ((-\sqrt{3}+1)/4, (-\sqrt{3}-1)/4) = B$,

$(\cos(3\pi/4)/\sqrt{2}, \sin(3\pi/4)/\sqrt{2}) = (-1/2, 1/2) = C$.

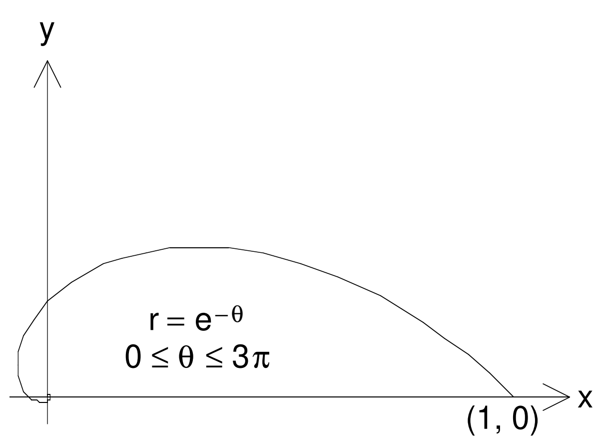
See graph above and to the right.

44. If $r = e^{-\theta}$ then $\frac{dr}{d\theta} = -e^{-\theta}$. The length of the portion of the graph of $r = e^{-\theta}$ with $0 \le \theta \le 3\pi$ is

$L = \int_0^{3\pi} \sqrt{(e^{-\theta})^2 + (-e^{-\theta})^2}\, d\theta =$

$= \int_0^{3\pi} \sqrt{2e^{-2\theta}}\, d\theta = \int_0^{3\pi} \sqrt{2}\, e^{-\theta}\, d\theta =$

$= -\sqrt{2}\, e^{-\theta}\Big]_0^{3\pi} = \sqrt{2}\left(1 - e^{-3\pi}\right)$.



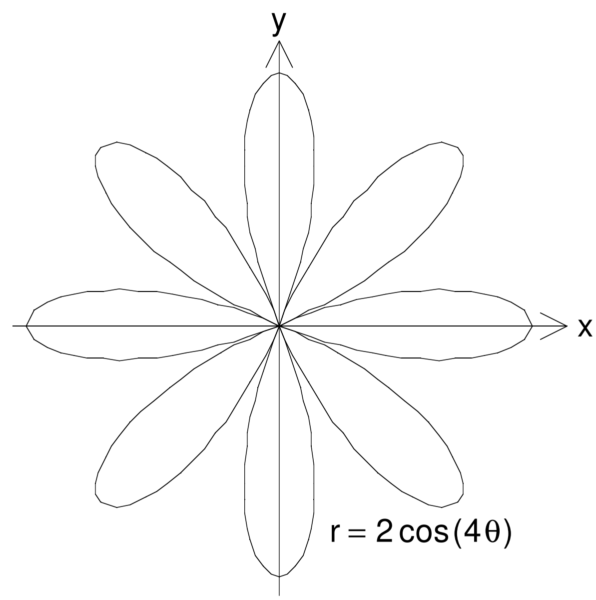
For Exercise 44

See graph to the right. As θ increases, $r = e^{-\theta}$ decreases dramatically, so only the portion with $0 \le \theta \le 3\pi/2$ is clearly

18. $r = 2\cos(4\theta)$ has eight loops.
One loop occurs when $-\pi/8 \le \theta \le \pi/8$.
The area inside that loop is

$$A = \int_{-\pi/8}^{\pi/8} \frac{1}{2}[2\cos(4\theta)]^2\,d\theta = \int_{-\pi/8}^{\pi/8} 2\cos^2(4\theta)\,d\theta = \int_{-\pi/8}^{\pi/8}[(1+\cos(8\theta)]\,d\theta = \left[\theta + \frac{1}{8}\sin(8\theta)\right]\Bigg]_{-\pi/8}^{\pi/8} = \frac{\pi}{4}.$$

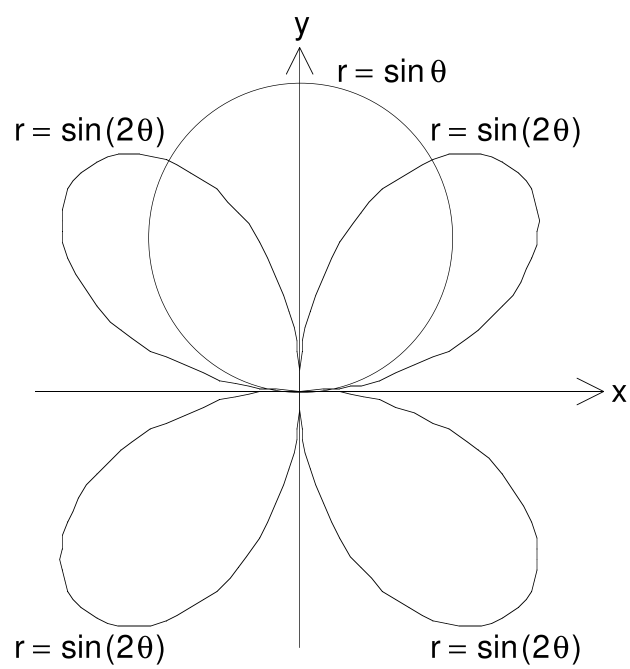
See graph to the right.



For Exercise 18

28. $r = \sin(2\theta)$ has four loops for $0 \le \theta \le 2\pi$, one occurring for each of $0 \le \theta \le \pi/2$, $\pi/2 \le \theta \le \pi$, $\pi \le \theta \le 3\pi/2$, and $3\pi/2 \le \theta \le 2\pi$. The second and fourth loops involve negative values for r, while the first and third involve positive r values. The circle $r = \sin\theta$ is traced out twice for $0 \le \theta \le 2\pi$, the entire graph occurring in the first and second quadrants. So in addition to meeting at the origin, the two graphs also meet in the interiors of the first and second quadrants, but the second quadrant intersection uses a fourth quadrant value for θ for the graph of $r = \sin(2\theta)$.

If we solve $\sin(2\theta) = \sin\theta$, or in other words if we solve $2\sin\theta\cos\theta = \sin\theta$, we obtain $\sin\theta = 0$ or $\cos\theta = 1/2$, so in addition to the intersections at the origin we also have intersections when $\theta = \pi/3$ and when $\theta = 5\pi/3$, at the points with rectangular coordinates $(\sqrt{3}/4, 3/4)$ and $(-\sqrt{3}/4, 3/4)$.



For Exercise 28

For the interval $0 < \theta < \pi/3$, and also for $5\pi/3 < \theta < 2\pi$, the curve closer to the origin is the circle $r = \sin\theta$.
For the interval $\pi/3 < \theta < \pi/2$, and also for the interval $3\pi/2 < \theta < 5\pi/3$, the curve closer to the origin is $r = \sin(2\theta)$.

The first- and the second-quadrant portions have equal areas. Thus the total area is

$$A = 2\int_0^{\pi/3} \frac{1}{2}(\sin\theta)^2\,d\theta + 2\int_{\pi/3}^{\pi/2} \frac{1}{2}(\sin(2\theta))^2\,d\theta) = \int_0^{\pi/3}\sin^2\theta\,d\theta + \int_{\pi/3}^{\pi/2}\sin^2(2\theta)\,d\theta =$$

$$= \int_0^{\pi/3} \frac{1-\cos(2\theta)}{2}\,d\theta + \int_{\pi/3}^{\pi/2} \frac{1-\cos(4\theta)}{2}\,d\theta = \left(\frac{2\theta - \sin(2\theta)}{4}\right)\Bigg]_0^{\pi/3} + \left(\frac{4\theta - \sin(4\theta)}{8}\right)\Bigg]_{\pi/3}^{\pi/2} =$$

$$= \frac{\pi}{4} - \frac{3\sqrt{3}}{16}.$$

MATHEMATICS 152 98-2 Solutions for Assignment 10

2. The area bounded by $r = e^{\theta}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ is $A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}(e^{\theta})^2 d\theta =$

$= \int_{-\pi/2}^{\pi/2} \frac{1}{2} e^{2\theta} d\theta = \frac{1}{4} e^{2\theta}\Big]_{-\pi/2}^{\pi/2} = \frac{1}{4}[e^{\pi} - e^{-\pi}] = \frac{e^{2\pi} - 1}{4e^{\pi}} = \frac{1}{2}\sinh \pi.$

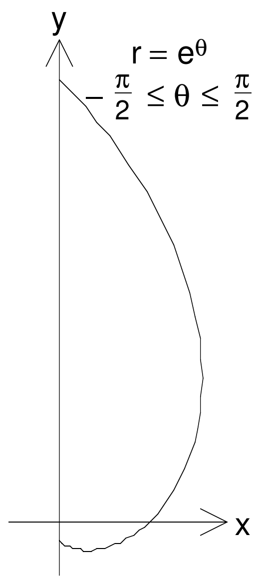
See graph below and to the left.

6. The area bounded by $r = \cos(3\theta)$, $-\frac{\pi}{12} \le \theta \le \frac{\pi}{12}$ is $A = \int_{-\pi/12}^{\pi/12} \frac{1}{2}[\cos(3\theta)]^2 d\theta =$

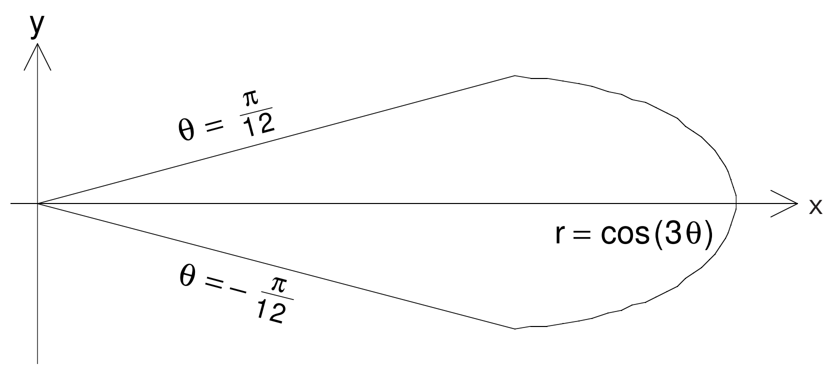
$= \int_{-\pi/12}^{\pi/12} \frac{1}{2}\cos^2(3\theta)\, d\theta = \int_{-\pi/12}^{\pi/12} \frac{1}{4}[1 + \cos(6\theta)]\, d\theta =$

$= \frac{1}{4}\Big[\theta + \frac{1}{6}\sin(6\theta)\Big]\Big]_{-\pi/12}^{\pi/12} = \frac{1}{4}\Big[\frac{\pi}{6} + \frac{1}{3}\Big] = \frac{\pi}{24} + \frac{1}{12}.$

See graph below and to the right.



For Exercise 2



For Exercise 6

8. $r = 4(1 - \cos\theta)$ is a cardioid, traced out once for $0 \le \theta \le 2\pi$, with enclosed area

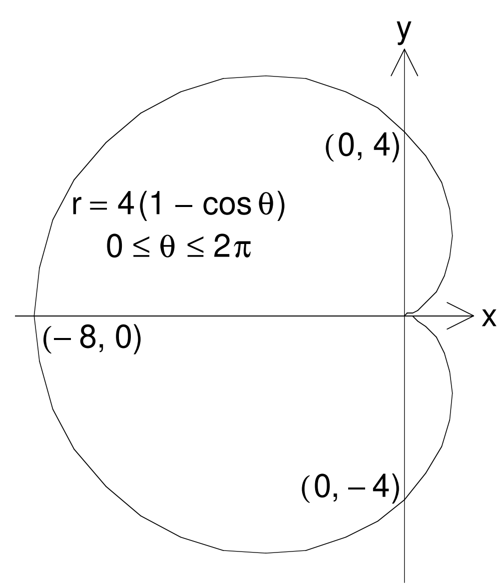
$A = \int_0^{2\pi} \frac{1}{2}[4(1 - \cos\theta)]^2 d\theta =$

$= \int_0^{2\pi} 8[1 - 2\cos\theta + \cos^2\theta]\, d\theta =$

$= \int_0^{2\pi} [12 - 16\cos\theta + 4\cos(2\theta)]\, d\theta =$

$= [12\theta - 16\sin\theta + 2\sin(2\theta)]\Big]_0^{2\pi} = 24\pi.$

See graph to the right.



For Exercise 8