24. For $p \le 0$, $\frac{(\ln n)^p}{n} \ge \frac{(\ln(n+1))^p}{n+1}$ for all $n \ge 1$, and $\lim_{n\to\infty} \frac{(\ln n)^p}{n} = 0$.

So $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(\ln n)^p}{n}$ converges by the alternating series test when $p \le 0$.

In fact the series also converges for all values of $p > 0$.

By L'Hospital's Rule, $\lim_{x\to\infty} \frac{(\ln x)^p}{x} = \lim_{x\to\infty} \frac{p\,(\ln x)^{p-1}\cdot x^{-1}}{1} = p \lim_{x\to\infty} \frac{(\ln x)^{p-1}}{x}$.

The exponent on $(\ln x)$ decreases by 1, and a constant coefficient p appears.

Now if $(p-1) \le 0$, we observed that $\lim_{x\to\infty} \frac{(\ln x)^{p-1}}{x} = 0$ in the first part of the exercise.

If $(p-1) > 0$, repeat the argument to see that $\lim_{x\to\infty} \frac{(\ln x)^p}{x} = p(p-1) \lim_{x\to\infty} \frac{(\ln x)^{p-2}}{x}$.

Continuing until the exponent has finally been reduced to 0 or less, we see that $\lim_{x\to\infty} \frac{(\ln x)^p}{x} = 0$ regardless of how large p may be! Changing from a real variable x to an integer variable n, $\lim_{n\to\infty} \frac{(\ln n)^p}{n} = 0$ for all values of p. Although for $p > 0$, the function $\frac{(\ln x)^p}{x}$ is **not** decreasing for **all** $x \ge 1$, it **is** decreasing for $x \ge e^p$ (take the first derivative to see why). So by chopping off the first part of the series (where the terms may be temporarily increasing in size) we can use the alternating series test on the rest of the series to see that it will converge regardless of the size of p. Once we know that, we can put the early terms back, changing the final sum but not the convergence status of the series.

26. To approximate $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$ with error less than 0.001, we use the partial sum $s_n = \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i^4}$ with $\frac{1}{(n+1)^4} < 0.001$. This first occurs with $n = 5$.

Since $s_5 = \frac{12280111}{12960000} \approx 0.947539429$, the desired approximation is 0.948.

30. To approximate $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}$ to four decimal places the error must be less than 0.00005. We use $s_n = \sum_{i=0}^{n} \frac{(-1)^i}{(2i)!}$ with $\frac{1}{[2(n+1)]!} < 0.00005$. This first occurs with $n = 3$. Since $s_3 = \frac{389}{720} \approx 0.540277777$, the desired approximation is 0.5403.

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2. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series so $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ is not absolutely convergent.

The conditions of the alternating series test are met, so the series is conditionally convergent.

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2. $-5 - \frac{5}{2} + \frac{5}{5} - \frac{5}{8} + \frac{5}{11} - \frac{5}{14} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\cdot 5}{3n-4}$ converges.

Except for the first term, the signs of the terms alternate; $\frac{5}{3n-4} \geq \frac{5}{3(n+1)-4}$ for $n \geq 1$; and $\lim_{n\to\infty} \frac{5}{3n-4} = 0$.

4. $\frac{1}{\ln 2} - \frac{1}{\ln 3} + \frac{1}{\ln 4} - \frac{1}{\ln 5} + \frac{1}{\ln 6} - \cdots = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ converges.

The signs of the terms alternate, $\frac{1}{\ln n} \geq \frac{1}{\ln(n+1)}$ for $n \geq 2$, and $\lim_{n\to\infty} \frac{1}{\ln n} = 0$.

8. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n \ln n}$ converges by the alternating series test.

The signs of the terms alternate, $\frac{1}{n \ln n} \geq \frac{1}{(n+1)\ln(n+1)}$ for $n \geq 2$, and $\lim_{n\to\infty} \frac{1}{n \ln n} = 0$.

10. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2+1}$ diverges.

Although it is an alternating series, the other two conditions are not met, since $\frac{n^2}{n^2+1} < \frac{(n+1)^2}{(n+1)^2+1}$, and $\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 \neq 0$ so that $\lim_{n\to\infty} (-1)^n \frac{n^2}{n^2+1}$ does not exist.

16. $\sum_{n=1}^{\infty} \frac{\sin \frac{n\pi}{2}}{n!} = \frac{1}{1!} + \frac{0}{2!} - \frac{1}{3!} - \frac{0}{4!} + \frac{1}{5!} + \frac{0}{6!} - \frac{1}{7!} - \frac{1}{8!} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)!}$ converges.

It is an alternating series with $\frac{1}{(2k-1)!} \geq \frac{1}{(2k+1)!}$ for all $k \geq 1$, and $\lim_{n\to\infty} \frac{1}{(2k-1)!} = 0$.

22. For $p > 0$, $\frac{1}{n^p} \geq \frac{1}{(n+1)^p}$ and $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

Thus $\sum_{k=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ converges by the alternating series test.

For $p = 0$, $\lim_{n\to\infty} \frac{1}{n^p} = \lim_{n\to\infty} \frac{1}{n^0} = \lim_{n\to\infty} \frac{1}{1} = 1 \neq 0$, so $\lim_{n\to\infty} \frac{(-1)^{n-1}}{n^p}$ does not exist, and worse yet for $p < 0$, $\lim_{n\to\infty} \frac{1}{n^p} = +\infty$ and $\lim_{n\to\infty} \frac{(-1)^{n-1}}{n^p}$ does not exist.

Thus $\sum_{k=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ diverges when $p \leq 0$.