MATHEMATICS 152 98-2 Solutions for Assignment 4

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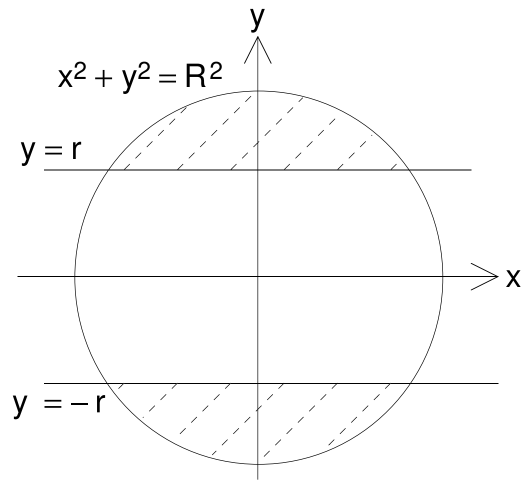
70. If the sphere is formed by rotating the circular disk $x^2 + y^2 \le R^2$ in 3-space about the x-axis and the cylindrical hole has axis of symmetry the x-axis, then the shell radius varies from $y = r$ to $y = R$. At radius y, the shell meets the sphere at $x = \pm\sqrt{R^2 - y^2}$, so the shell length is $2\sqrt{R^2 - y^2}$.

$$V = \int_r^R 2\pi y \cdot 2\sqrt{R^2 - y^2}\, dy =$$

$$= -\frac{4\pi}{3}(R^2 - y^2)^{3/2}\Big]_r^R = \frac{4\pi}{3}(R^2 - r^2)^{3/2}.$$

See graph to the right.



For Exercise 70

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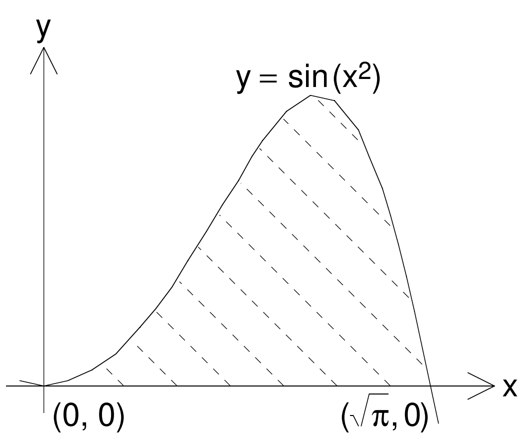
4. $V = \int_0^{\sqrt{\pi}} 2\pi x \sin(x^2)\, dx = -\pi \cos(x^2)\Big]_0^{\sqrt{\pi}} = 2\pi$. See graph below and to the left.

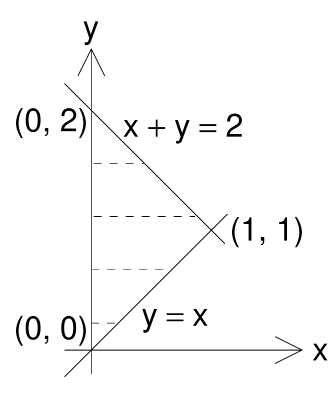
14. The lines $y = x$ and $x + y = 2$ meet at $(1, 1)$.

$$V = \int_0^1 2\pi y \cdot y\, dy + \int_1^2 2\pi y \cdot (2 - y)\, dy = \frac{2}{3}\pi y^3\Big]_0^1 + \Big[2\pi y^2 - \frac{2}{3}\pi y^3\Big]\Big]_1^2 = 2\pi.$$

This volume can be found with only one integration, using the method of washers.
See graph below and to the right.



For Exercise 4



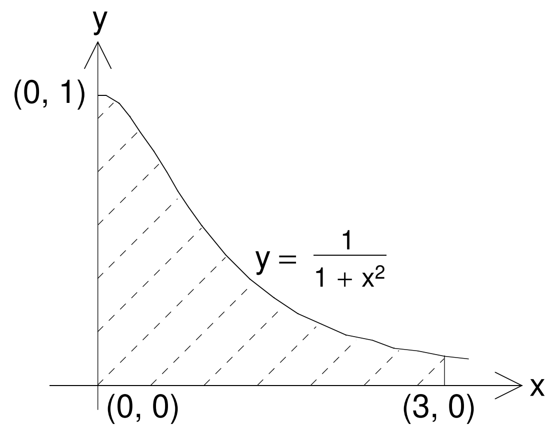
For Exercise 14

22. Using the method of cylindrical shells, the volume is $V = \int_0^3 2\pi x \frac{1}{1+x^2}\,dx$. Actually this is easy to evaluate.

$\int_0^3 2\pi x \frac{1}{1+x^2}\,dx = \pi \ln(1+x^2)\Big]_0^3 = \pi \ln 10.$

It is not too hard to do this exercise using disks, but we need to handle $0 \le y \le 0.1$ and $0.1 \le y \le 1$ separately because the radius is different in these two intervals. I leave the details to you.

See graph to the right.



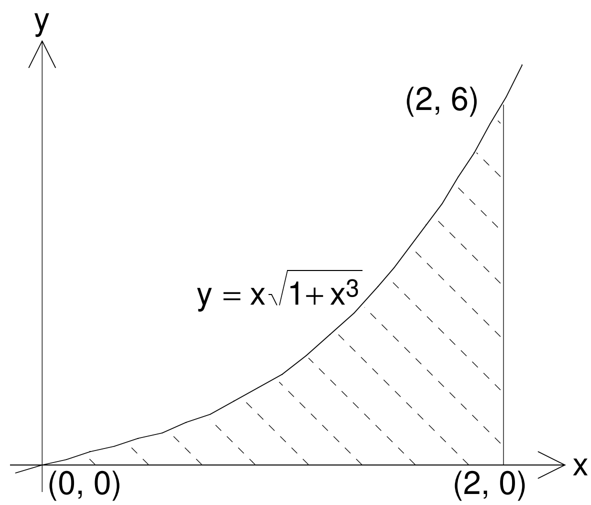
For Exercise 22

36. Using cylindrical shells,

$V = \int_0^2 2\pi x \cdot x\sqrt{1+x^3}\,dx =$

$= \frac{4\pi}{9}(1+x^3)^{3/2}\Big]_0^2 = \frac{104\pi}{9}.$

See graph to the right.



For Exercise 36

Could this volume be found using the method of washers? In order to do this we would have to evaluate the integral $V = \int_0^6 \pi[2^2 - (F(y))^2]dy$, where the inner radius $x = F(y)$ is obtained by solving the equation $y = x\sqrt{1+x^3}$ for x in terms of y. For positive values of y this equation is equivalent to the equation $y^2 = x^2 + x^5$. Some fifth degree equations can be solved, but there is no general method for solving all of them. Since solving this equation seems not to be a feasible procedure, the method of washers is not practical for this exercise, unless one resorts to Newton's Method or some other way of approximating the solution. In that case, with approximate numerical values for $x = F(y)$, $0 \le y \le 6$, numerical integration techniques would be required to evaluate the integral $\int_0^6 \pi[2^2 - (F(y))^2]dy$.

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In all the Exercises below, g is the acceleration due to gravity at the surface of the earth (approximately $9.8\ \text{m/s}^2$).

2. The force is a constant $F = 60g$ N ≈ 588 N.

The work done is $W = \int_0^2 60g\,dy = 60gy\Big]_0^2 = 120g \approx 1176$ J.

6. The spring stiffness is $k = \dfrac{25}{0.3-0.2} = 250$ N/m so the force is $F = kx = 250x$ N.

The work done is $W = \int_0^{0.05} 250x\,dx = 125x^2\Big]_0^{0.05} = 0.3125$ J.

12. Each portion of the cable y ft below the top of the building is lifted y ft if $0 \le y \le 10$ but is lifted 10 ft if $10 \le y \le 40$. The linear density is $\dfrac{60}{40} = 1.5$ lb/ft.

The work done is $W = \int_0^{10} 1.5y\,dy + \int_{10}^{40} 1.5\cdot 10\,dy = 0.75y^2\Big]_0^{10} + 15y\Big]_{10}^{40} = 525$ ft lb.

14. Water is leaking out at a rate of $\dfrac{0.2\text{ lb/s}}{2\text{ ft/s}} = 0.1$ lb/ft so the force exerted when the bucket has been pulled up y ft is $(44 - 0.1y)$ lb.

The work done is $W = \int_0^{80} (44-0.1y)\,dy = (44y - 0.05y^2)\Big]_0^{80} = 3200$ ft lb.

18. A slab of water y m above the centre of the tank has width $2\sqrt{1.5^2 - y^2}$ m and thus has area $12\sqrt{1.5^2-y^2}$ m^2; it must be lifted $(2.5 - y)$ m.

The work done is $W = \int_{-1.5}^{1.5} 12\cdot 1000g\sqrt{1.5^2-y^2}\,(2.5-y)\,dy =$

$$= \int_{-1.5}^{1.5}\left(30000g\sqrt{1.5^2-y^2} - 12000gy\sqrt{1.5^2-y^2}\right)dy =$$

$$= \left(15000gy\sqrt{1.5^2-y^2} + 33750g\sin^{-1}\frac{y}{1.5} + 4000g\left(1.5^2-y^2\right)^{3/2}\right)\Bigg]_{-1.5}^{1.5} = 33750g\pi \approx 1.04\times 10^6\text{J}.$$

You can make a very reasonable objection to this question: how did I ever find that strange antiderivative? (Presumably you have already differentiated it and verified that it is correct!) We'll address that question later, but for now it is enough to see that the question can be **avoided**. Since $x^2 + y^2 = 1.5^2$ is the equation of a circle of radius 1.5, $\int_{-1.5}^{1.5} 30000g\sqrt{1.5^2-y^2}\,dy$ is $15000g$ times the area of a circular disk of radius 1.5, or $33750g\pi$. (Integrate $x = \sqrt{1.5^2-y^2}$ with respect to y to give the area of the right half of the disk.) But $\int_{-1.5}^{1.5}\left(-12000gy\sqrt{1.5^2-y^2}\right)dy = 4000g(1.5^2-y^2)^{3/2}\Big]_{-1.5}^{1.5} = 0$ is an easy integration, so the entire integral is $33750g\pi$.

22. This time the density is only $920\ \mathrm{kg/m^3}$ and a slab of oil in the tank can have height above the centre of the tank varying from -1.5 m to 0 m.

The work done is $W = \int_{-1.5}^{0} 12\cdot 920\, g\sqrt{1.5^2 - y^2}\,(2.5 - y)\,dy =$

$= \int_{-1.5}^{0} \left(27600\, g\sqrt{1.5^2 - y^2} - 11040\, g\, y\sqrt{1.5^2 - y^2}\right)dy =$

$= \left(13800\, g\, y\sqrt{1.5^2 - y^2} + 31050\, g \sin^{-1}\frac{y}{1.5} + 3680\, g\left(1.5^2 - y^2\right)^{3/2}\right)\Big]_{-1.5}^{0} =$

$= 31050g\,\frac{\pi}{2} + 3680\cdot 1.5^3 g = (15525\pi + 12420)g \approx 6.00\times 10^5$ J.

As in Exercise 18, to avoid antidifferentiating think of $\int_{-1.5}^{0} 27600\, g\sqrt{1.5^2 - y^2}\,dy$ as $6900g$ times the area of a circular disk of radius 1.5, or $15525\, g\,\pi$. (Integrate $x = \sqrt{1.5^2 - y^2}$ with respect to y to give the area of the lower right quarter of the disk.)

As before, $\int_{-1.5}^{0} \left(-11040\, g\, y\sqrt{1.5^2 - y^2}\right)dy = 3680\, g\,(1.5^2 - y^2)^{3/2}\Big]_{-1.5}^{0} = 12420\, g$ is an easy integration, so the entire integral is $(15525\pi + 12420)g$.

24. $PV^{1.4} = k$ hence $P = kV^{-1.4}$. When $V = 100\ \mathrm{in^3}$, $P = 160\ \mathrm{lb/in^2}$ hence $k = 160\cdot 100^{1.4} = 1.6\times 10^{4.8}\ \mathrm{lb\ in^{2.2}}$.

The work done is $W = \int_{100}^{800} 160\cdot 100^{1.4}\, V^{-1.4}\, dV = (-400\cdot 100^{1.4}\, V^{-0.4})\Big]_{100}^{800} =$

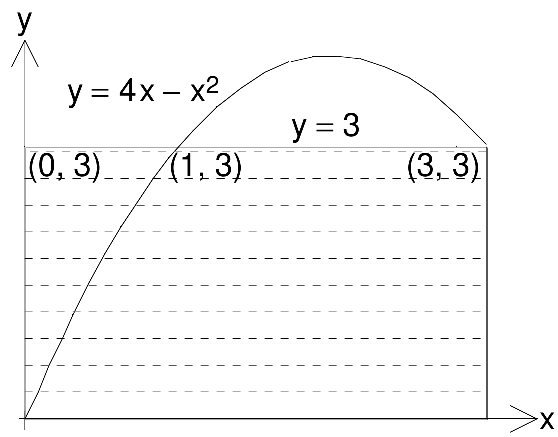
$= 4\cdot 10^{4.8}[100^{-0.4} - 800^{-0.4}] = 40000\cdot[1 - 2^{-1.2}] \approx 2.26\times 10^4$ in lb or 1.88×10^3 ft lb.

26. The force exerted is $F = 10^3\,\frac{GM}{r^2}$ N, where G is the gravitational constant measured in $(\mathrm{Nm^2})/\mathrm{kg^2} = \mathrm{m^3/(kg s^2)}$, M is the mass of the earth measured in kg, and r is the distance from the centre of the earth to the satellite measured in m. Let R be the radius of the earth measured in m.

The work done is $W = \int_{R}^{R+10^6} 10^3 GM r^{-2}\, dr = -10^3 GM r^{-1}\Big]_{R}^{R+10^6} =$

$= 10^3 GM\left[\frac{1}{R} - \frac{1}{R + 10^6}\right] = 10^9 GM R^{-1}(R + 10^6)^{-1}$. But $G \approx 6.67\times 10^{-11}\ (\mathrm{Nm^2})/\mathrm{kg^2}$, $R \approx 6.37\times 10^6$ m, and $M \approx 5.98\times 10^{24}$ kg, so $W \approx 8.50\times 10^9$ J.

2. $f_{ave} = \frac{1}{\pi - 0}\int_0^{\pi} \sin x \, dx =$
$= \frac{1}{\pi}(-\cos x)\Big]_0^{\pi} = \frac{2}{\pi}.$



For Exercise 8

6. $f_{ave} = \frac{1}{9 - 4}\int_4^{9} \sqrt{x}\, dx =$
$= \frac{1}{5}\left(\frac{2}{3}x^{3/2}\right)\Big]_4^9 = \frac{38}{15}.$

8. (a) $f_{ave} = \frac{1}{3 - 0}\int_0^{3} (4x - x^2)\, dx =$
$= \frac{1}{3}\left(2x^2 - \frac{1}{3}x^3\right)\Big]_0^3 = 3.$

(b) If $f(c) = f_{ave} = 3$, $4c - c^2 = 3$, so $c^2 - 4c + 3 = 0$, $(c - 1)(c - 3) = 0$, and $c = 1$ or $c = 3$.

(c) See graph above and to the right.

12. If $f_{ave} = \frac{1}{b - 0}\int_0^{b} (2 + 6x - 3x^2)\, dx = \frac{1}{b}(2x + 3x^2 - x^3)\Big]_0^b = 2 + 3b - b^2 = 3$ then $b^2 - 3b + 1 = 0$ and $b = \frac{3 \pm \sqrt{5}}{2}$.

14. $T_{ave} = \frac{1}{5 - 0}\int_0^{5} 4x\, dx = \frac{2}{5}x^2\Big]_0^5 = 10\ °C.$

16. If $s = \frac{1}{2}gt^2$ then $v = gt$ so $v_T = gT$. Notice $t = (2s/g)^{1/2}$, so $v = (2gs)^{1/2}$.
$v_{ave} = \frac{1}{T - 0}\int_0^{T} v(t)\, dt = \frac{1}{T}\int_0^{T} gt\, dt = \frac{1}{T}\frac{1}{2}gt^2\Big]_0^T = \frac{1}{2}gT = \frac{1}{2}v_T$, if we average the velocity with respect to time.
$v_{ave} = \frac{1}{(gT^2/2) - 0}\int_0^{(gT^2/2)} v(t)\, ds = \frac{2}{gT^2}\int_0^{(gT^2/2)} (2gs)^{1/2}\, ds = \frac{2}{gT^2}\cdot\frac{1}{3g}(2gs)^{3/2}\Big]_0^{(gT^2/2)} =$
$= \frac{2}{3}gT = \frac{2}{3}v_T$, if we average the velocity with respect to position.

18. $v_{ave} = \frac{1}{R - 0}\int_0^{R} \frac{P}{4\eta l}(R^2 - r^2)\, dr = \frac{1}{R}\cdot\frac{P}{4\eta l}\cdot\left(R^2 r - \frac{1}{3}r^3\right)\Big]_0^R = \frac{1}{6}\frac{PR^2}{\eta l}.$
Notice $v(r) = \frac{P}{4\eta l}(R^2 - r^2)$ is decreasing for $0 \le r \le R$, so $v_{max} = v(0) = \frac{1}{4}\frac{PR^2}{\eta l}$ and $v_{ave} = \frac{2}{3}v_{max}$.