ZEROS OF PARTIAL SUMS OF THE RIEMANN ZETA-FUNCTION

PETER BORWEIN, GREG FEE, RON FERGUSON, AND ALEXA VAN DER WAALL

1. Introduction

The following is a picture of the first 10,000 suitably normalized zeros of the fifth partial sum of the Riemann zeta-function. (Precise definitions are given in later sections.) The picture is compelling – it suggests a phenomenon in need of an

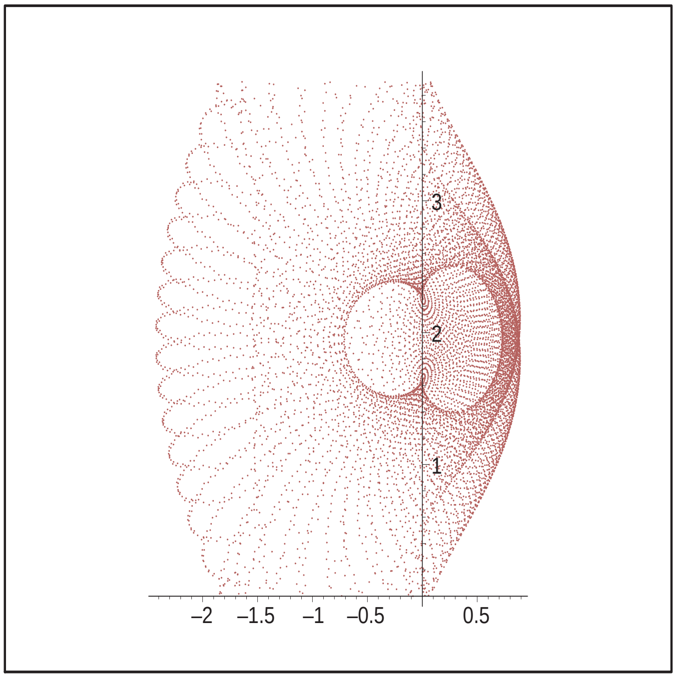

Figure 1. 10,000 normalized zeros of $\zeta_5(s)$

explanation - and this is indeed part of both the genesis and the purpose of this paper.

A second part of the genesis of the paper is a project to write a robust zero finder, in Maple, for virtually any analytic function on virtually any region. This is a hard problem that will be discussed elsewhere. Finding a few zeros is easy;

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reliably finding all zeros is much harder. Even special purpose pictures, like those in this paper, require some effort to produce. Off the shelf software will probably not suffice.

Of course any number theorist will first apply these methods to the Riemann zeta-function in the hope of shedding a little more light on what number theorists certainly believe to be the most important unsolved problem in mathematics.

Section 2 gives a brief introduction to the Riemann zeta function. For a more comprehensive treatment the recent publication [Con03] is recommended.

Section 3 discusses previous results for partial sums. Figure 2, made possible by our zero finder, indicates that, even though the partial sums of $\zeta_N(s)$ do not converge for $\Re(s) \leq 1$, there is some range in the complex plane where the zeros of the partial sum do approximate the zeros of $\zeta(s)$. Figure 3 gives a clearer indication that zeros of partial sums of the alternating series $\eta(s)$ converge to zeros of the function itself. Theorem 3.1 improves previously published results on upper and lower bounds for the real part of zeros of partial sums for $\zeta(s)$ and $\eta(s)$. Refining this, in Theorem 4.7 we obtain a precise lower bound on the real parts of zeros of $\zeta_N(s)$ for N prime.

Since $\zeta(s)$ encodes so much information about prime numbers, it is natural to expect that a study of zeros of a partial sum will involve some study of the prime numbers involved in the sum. The first part of Section 4 studies the cases where two primes are involved. The zeros of $\zeta_3(s)$ are shown to be intersection points of two periodic functions, with periods $2\pi i/\log(2)$ and $2\pi i/\log(3)$. A careful analysis shows precisely which intersection points are zeros of ζ_3 and their density as a function of $\Im(s)$. Theorem 4.5 gives a more general result for finite exponential sums involving only two primes, stating that all zeros lie on a periodic curve, where the number of choices for this period is infinite.

The second part of Section 4 in an extension to where more primes are involves. The main result is Theorem 4.6, which shows how the discrete zeros of an finite exponential sum can be related to the zeros of a multiply periodic function. In particular, precise bounds for the distribution of these zeros may be determined using methods of calculus.

2. Background

Early studies of the function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s \in \mathbb{C} \tag{1}$$

concerned its behavior for real s. Euler pointed that this has a representation as a product

$$\zeta(s) = \prod_{p \text{ prime}} (1 - \frac{1}{p^s})^{-1},$$

valid for $s \in \mathbb{C}$ with $\Re(s) > 1$, first giving a hint of its importance in the study of prime numbers. It was Riemann, however, who, in his fundamental paper of 1859, outlined the analytic properties of what is now known as the *Riemann zeta function*. Writing s in the standard form

$$s = \sigma + it,$$

where σ and t are the real and imaginary parts of s, respectively, it is easily seen that the above series converges absolutely for $\sigma > 1$. Riemann showed that $\zeta(s)$ has an analytic extension to a meromorphic function on $\mathbb{C}$ having a single simple pole at $s = 1$. Moreover, he proved the surprising functional equation

$$\xi(s) := \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(s/2\right)\zeta(s) = \xi(1-s),$$

where $\Gamma(s)$ is the usual Gamma function. Using Euler's product formula, we derive that $\zeta(s)$ has no zeros for $\sigma > 1$. The functional equation then shows that the only zeros of $\zeta(s)$ is the left half plane $\sigma < 0$ coincide with the poles of $\Gamma(s/2)$, the so called *trivial zeros* at $s = -2, -4, -6, \ldots$. All other zeros must lie in the *critical strip* $0 \leq \sigma \leq 1$. Riemann asserted furthermore that the number of these roots whose imaginary parts are between 0 and $T > 0$ is approximately

$$\frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi}, \tag{2}$$

with relative error term $O(1/T)$. Riemann stated that the same estimate should hold for all roots with real parts 1/2, leading to the stronger conjecture that all roots of the zeta-function in the critical strip lie on the *critical line* $\sigma = 1/2$, the famous *Riemann Hypothesis*. In 1905 Von Mangoldt proved the estimate (2) is correct, proving the weaker error estimate of $O(\log(T))$, see [Tit86, CH IX].

The complex line $\{z \in \mathbb{C} : \Re(s) = 1/2\}$ is known as the *critical line*. The Riemann Hypothesis was one of the 23 problems given by Hilbert in 1900 for mathematicians to work on in the 20th century. It remains a problem for mathematicians in the 21st century. It is know that there are at least $KT\log(T)$ such zeros for some constant $K > 0$ as T tends to ∞, see [Tit86, §10.9] or [Edw01]. Also Conrey proved [Con89] that more than 40% of the zeros of the Riemann zeta-function are on the critical line. Recent computations (see **www.zetagrid.net/zeta/rh.html**) have confirmed that the first 100 billion zeros with $t > 0$ are indeed on the critical line.

3. Zeros of partial sums

Since the series (1) does not converge for $\sigma \leq 1$, it it is difficult to picture, on the face of it, what relationship, if any, would exist between the zeros of the *truncated zeta-function*

$$\zeta_N(s) := \sum_{n=1}^{N} \frac{1}{n^s}$$

and $\zeta(s)$ in this half plane. The following plot, made possible through the application of the zero finder, illustrates a few points. What is first noticeable is a string of zeros of ζ_{211} near the critical line $\sigma = 1/2$. Spira remarks in [Spi66, §4] the equality

$$\begin{aligned}\zeta(s) \quad &= \quad \zeta_{N-1}(s) + N^{-s} + \frac{N^{1-s}}{s-1} + \\ &\quad + \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}\left(\prod_{j=0}^{2n-2}(s+j)\right) N^{1-s-2n} + R\end{aligned}$$

implies that $\zeta(s)$ is roughly approximated by $\zeta_N(s)$ near the critical line for $t < 2\pi N$, but t also large enough so that $N^{1/2}/t$ is 'small'. The strip of zeros is in

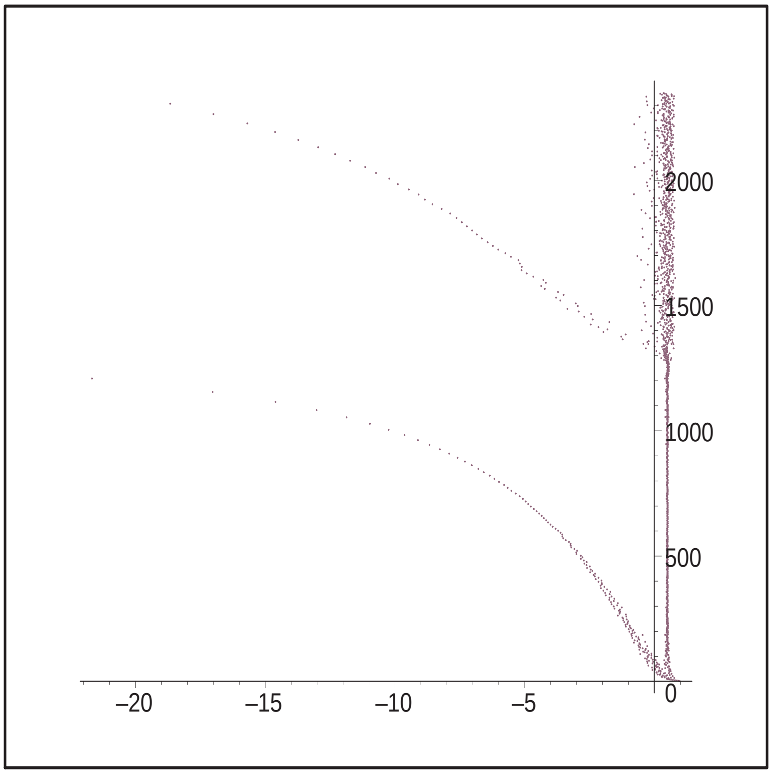

FIGURE 2. 2000 zeros of $\zeta_{211}(s)$

accordance with Spira's observation. Above $t = 2\pi N$, about 1326, in Figure 2, the zeros scatter more wildly.

In [Tur48], Turán does point to a possible relationship between the behavior of the roots of the partial sums of the zeta-function (1) and the Riemann Hypothesis. He proves in his Theorem II that the Riemann hypothesis is valid if there are positive numbers n_0 and K such that for $N > n_0$ the truncated zeta-function $\zeta_N(s)$ does not vanish in the half plane

$$\sigma \geq 1 + \frac{K}{\sqrt{N}}.$$

In [Mon83], however, Montgomery shows that for given $0 < c < \frac{4}{\pi} - 1$ and for N large enough the function ζ_N always has zeros in the half plane

$$\sigma > 1 + c\frac{\log\log(N)}{\log(N)},$$

making Turán's theorem vacuous. Still, it is interesting that connections can be made between zeros of $\zeta(s)$ in the critical strip and zeros of the partial sums $\zeta_N(s)$.

A result of Knopp ([Jen18, p. 236]) asserts that every point of the line $\sigma = 1$ is an accumulation point of the zeros of the functions ζ_N. Striking in Figure 2 are the two strings of zeros trailing to the left, both above and below the string of zeros near the critical line. The lower string originates in another region about the critical strip where the zeros scatter wildly. To fulfil Knopp's observation, this region must expand upward as N increases.

The comparison with partial sums in the critical strip is more direct for the *alternating zeta-function*, defined by

$$\eta(s) := \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^s},$$

in its region of convergence $\sigma > 0$ and then extended by analytic continuation. We derive that

$$\eta(s) = \left(1 - \frac{2}{2^s}\right) \zeta(s),$$

so all the zeros of $\zeta(s)$ are zeros of $\eta(s)$. In particular, $\zeta(s)$ and $\eta(s)$ have the same zeros within the critical strip. As well, $\eta(s)$ has the zeros of $1 - 2^{1-s}$, namely, the points

$$\omega_k := 1 + \frac{2k\pi i}{\log(2)}, \quad k \in \mathbb{Z}. \tag{3}$$

The *truncated alternating zeta-functions*

$$\eta_N(s) := \sum_{n=1}^{N} (-1)^{n+1} \frac{1}{n^s}, \quad N \in \mathbb{Z}_{>0}$$

converge in the plane $\sigma > 0$ to the alternating zeta-function as N tends to ∞. In particular they have roots 'close' to the ω_k's if N is sufficiently large. More specifically, the point

$$\omega_k + \frac{1}{4\log(2) \cdot \zeta(\omega_k)} (N+1)^{-2+\omega_k}$$

is the asymptotic expression of a root of η_{2N} that converges to ω_k for $N \to \infty$, see [Tur48, §7].

Consider now Figure 3 below. Above $t = 0$, we see two strings of zeros, one close to the critical line approximating the zeros of the zeta function, and the other close to $s = 1$ approximating the zeros of $1 - 2^{1-s}$, as expected. Above these the zeros scatter more wildly, as in Figure 2.

It has been proved in the papers [Lan31] and [Wil17] that the number M of zeros of ζ_N in the upper half plane that have imaginary part $< T$ satisfies

$$\left| M - \left\lfloor \frac{\log(N)T}{2\pi} \right\rfloor \right| \leq N. \tag{4}$$

This approximates the density of the zeros of $1 \pm 1/N^s$ whic occur one every distance of $2\pi/\log(N)$ up the imaginary axis, or a frequency of $\log(N)/2\pi$.

Using (2) and (3), the number of roots if $\eta(s)$ in the same region has the estimate

$$\frac{T}{2\pi} \log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi} + \frac{(\log(2))T}{2\pi}.$$

Differentiating with respect to T, we obtain a frequency of $\log(T/\pi)/2\pi$ for these roots as we move up the imaginary axis. These two frequencies match at $T = N\pi$, which for Figure 3 evaluates as $342.4\ldots$, approximately where this scattering begins. Above this, the number of zeros of the partial sum is insufficient to match those of $\eta(s)$, while below they are in excess, giving some explanation for the string of zeros trailing to the left at the bottom.

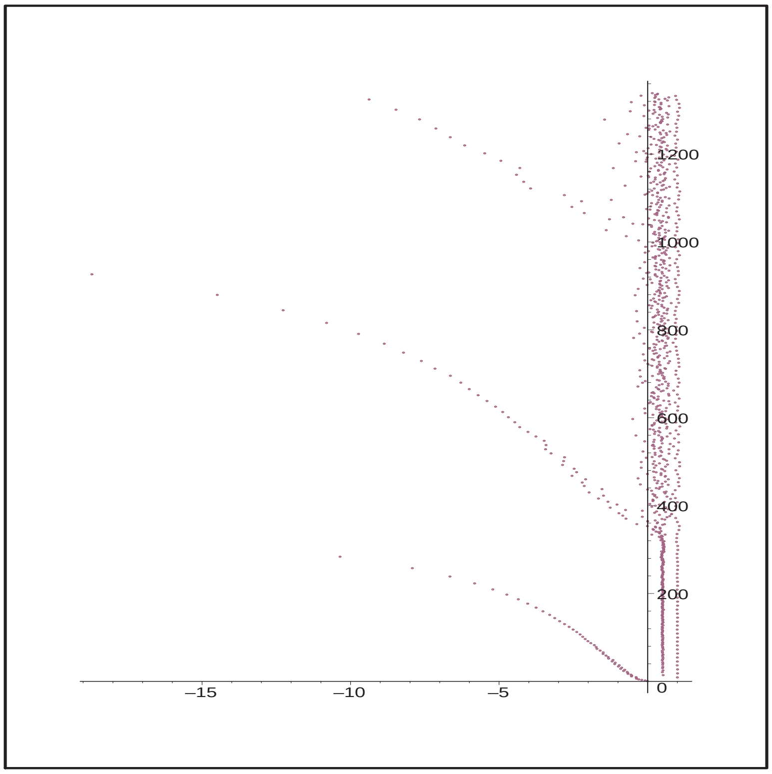

FIGURE 3. 1000 zeros of $\eta_{109}(s)$

The complete set of accumulation points of zeros of the truncated alternating zeta-functions consists of the line $\sigma = 0$, the roots of $\zeta(s)$ and the zeros ω_k, see [Tur48, §7]. This line of zeros appears to follow a smooth curve trailing to the left. Less noticeable is a slight scattering of these roots closer to the origin. This scattering must increase with N to account for $s = 0$ being a line of accumulation points.

Turán showed [Tur59] that the postions of zeros in the half-plane $\sigma > 1$ for the partial sums of the alternating zeta function also has relevance for the Riemann Hypothesis. Though his estimates in this case as well were proved redundant by Montgomery's work, it is interesting to obtain actual bounds for the real values of these roots.

It is proved in [Spi66] that all zeros of ζ_N have real parts

$$1 - N < \sigma < 1.85.$$

The following simple argument improves this.

Theorem 3.1. *Let s be a zero of ζ_N or η_N. Then s satisfies $\sigma \leq \alpha$, where α is the positive root of $\zeta(\sigma) = 2$. In particular one has $\sigma < 1.73$.*

Also s satisfies $\sigma \geq \beta_{_N}$, where $\beta_{_N}$ is the negative root of $\zeta_{_{N-1}}(\sigma) = N^{-\sigma}$.

Proof. Let s be a zero of ζ_N such that $\sigma > 1$. Then

$$\begin{aligned} 1 &= |-2^{-s} - 3^{-s} - \ldots - N^{-s}| \\ &\leq 2^{-\sigma} + 3^{-\sigma} + \ldots + N^{-\sigma} \\ &\leq \zeta(\sigma) - 1. \end{aligned}$$

As the functions in these inequalities are positive and decreasing in σ we need to find the positive solution of $\zeta(s) = 2$. This is $s = 1.728647239$.

Similarly,

$$\begin{aligned} |N^{-s}| = N^{-\sigma} &= |1 + 2^{-s} + 3^{-s} + \ldots + (N-1)^{-s}| \\ &\leq 1 + 2^{-\sigma} + 3^{-\sigma} + \ldots + (N-1)^{-\sigma}. \end{aligned}$$

Note that $\zeta_{N-1}(\sigma) = N^{-\sigma}$ has a single negative root, $\sigma = \beta_N$. For $\sigma < \beta_N$ the above inequality no longer holds.

These proofs hold in the same way for the zeros of η_N. □

For $\zeta_N(s)$, Turán shows in [Tur48, Theorem IV] that for N large enough $\zeta_N(s) \neq 0$ for

$$\sigma \geq 1 + 2(\log\log N)/\log(N),$$

so an improved upper bound should be possible in this case. However, the lower bound is closer to the truth. Indeed, for N prime, we show in Theorem 4.7 below that this is the actual bound.

An easy argument shows that $\beta_N = (N + o(N))\log(2)$. Writing $\beta_N = b_N N$, we have

$$N^{-b_N N} + \ldots + \left(\left(1 - \frac{2}{N}\right)^N\right)^{-b_N} + \left(\left(1 - \frac{1}{N}\right)^N\right)^{-b_N} = 1.$$

We note that $\lim_{N\to\infty}\left(1 - \frac{k}{N}\right)^N = e^k$. A limit $b_N \to b_\infty$ would imply

$$1 = e^{-b_\infty} + e^{-2b_\infty} + e^{-3b_\infty} + \ldots$$

giving $b_\infty = \log(2)$ as a first approximation. Computationally, the estimate $\beta_N \simeq (N - 3/2)\log(2)$ holds with surprising accuracy at $N = 5$ and with increasing accuracy to at least $N = 100,000$.

4. Zeros of small sums

4.1. **Single parameter curves.** Figure 4 shows the positions of the first 2000 zeros of

$$\zeta_3(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s}$$

in the upper half plane.

Although the zeros are not recurring in a completely regular pattern, their positioning does appear to have a semiperiodic nature. The algorithm used to locate these zeros used a homotopy. In its more general form, for ζ_N, starting from the known position of the zeros of the end terms

$$1 + \frac{1}{N^s},$$

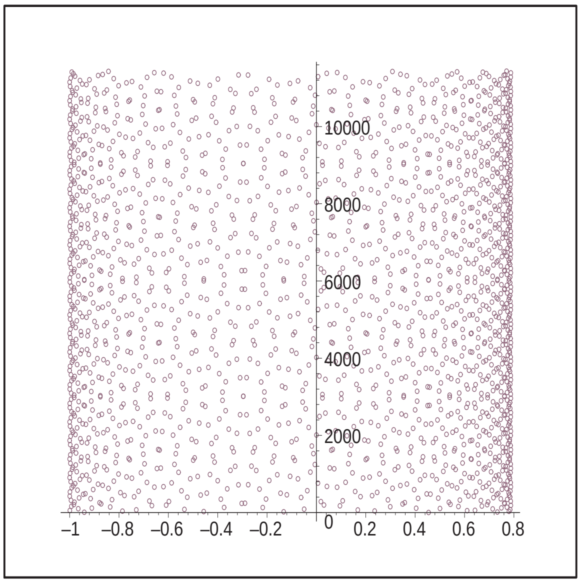

FIGURE 4. First 2000 zeros of $\zeta_3(s)$

at the points $\{2\pi k i/\log(N) : k \in \mathbb{N}\}$, Newton's method was used to find zeros along the path

$$1 + \frac{1}{N^s} + t\left(\sum_{n=2}^{N-1} \frac{1}{n^s}\right)$$

as t increased from 0 to 1. That this method worked so well in locating all of the zeros of $\zeta_N(s)$ up to heights tested, suggests that the error estimate in (4) above could be improved to $O(1)$. In any case, in such finite exponential sums, the largest integer in the expansion, N, is an indicator or the number of zeros to be found up to a height T in much the same way as the degree of a polynomial determines its number of zeros in the complex plane.

It is then natural to plot the difference between the zeros found for ζ_N and the starting values of each path. This was the inspiration for the plot on the left of Figure 5, where the imaginary parts of the zeros are reduced modulo $2\pi i/\log(3)$, the distance between zeros of $1 + 1/3^s$.

We now see a striking regularity.

An explanation is fairly simple. The equation $\zeta_3(s) = 0$ becomes two in the real and imaginary parts:

$$1 + 2^{-x}\cos(\log(2)y) + 3^{-x}\cos(\log(3)y) = 0 \tag{5}$$

$$2^{-x}\sin(\log(2)y) + 3^{-x}\sin(\log(3)y) = 0, \tag{6}$$

where $s = x+iy$. Using the trigonometric formula $\cos^2(\log(2)y)+\sin^2(\log(2)y) = 1$, we derive

$$1 + 2 \cdot 3^{-x}\cos(\log(3)y) + 9^{-x} = 4^{-x}. \tag{7}$$

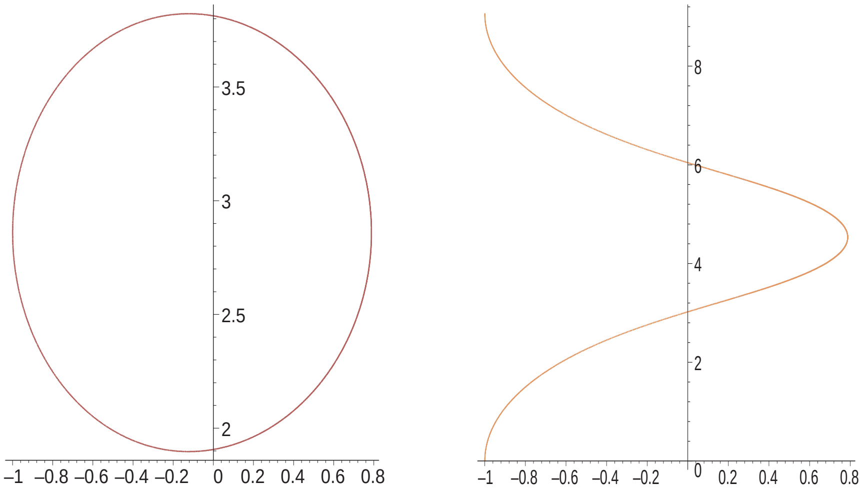

FIGURE 5. Zeros of $\zeta_3(s)$ reduced modulo $\frac{2\pi i}{\log(3)}$ and $\frac{2\pi i}{\log(2)}$

This is a curve with period $2\pi i/\log(3)$ in y which must contain all the zeros of $\zeta_3(s)$. Reducing the imaginary parts of the zeros modulo $2\pi i/\log(3)$, they still lie on the same curve, but all within a single period.

The same procedure may be applied to remove terms in $\log(3)y$, giving

$$1 + 2 \cdot 2^{-x} \cos(\log(2)y) + 4^{-x} = 9^{-x}. \tag{8}$$

Zeros reduced modulo $2\pi i/\log(2)$ lie on a period of this curve, as shown on the right of Figure 5.

We can use the curves (7) and (8) to obtain a good description of the zeros of $\zeta_3(s)$. Figure 6 below allow us to compare intersection points. The graph on

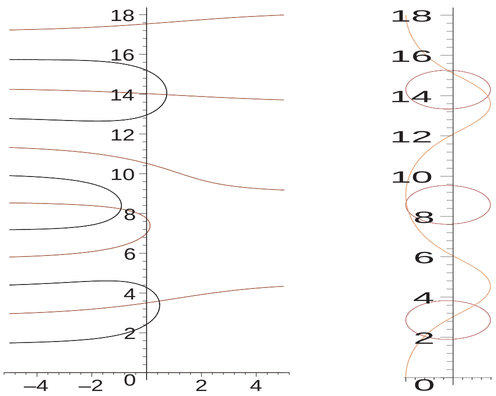

FIGURE 6. Comparison of intersection points

the left shows intersection points of the curves $\Re(\zeta_3(s)) = 0$ and $\Im(\zeta_3(s)) = 0$ as given in (5) above, these points being zeros of $\zeta_3(s)$. The graph on the right shows intersection points of the curves (7) and (8). These points include the zeros of $\zeta_3(s)$, but, as we can see in the comparison, not all of the intersection points are zeros of $\zeta_3(s)$. This graph does show that the real parts of of these zeros are bounded, something now evident from the graph on the left.

A more careful analysis is needed to show which intersection points of (7) and (8) are zeros of $\zeta_3(s)$ and to describe their distribution.

Lemma 4.1. *The zeros of $\zeta_3(s)$ are those points satisfying equations (7) and (8) for which $\sin(\log(3)y)$ and $\sin(\log(2)y)$ are of opposite sign.*

Proof. Any zero of $\zeta_3(s)$ must satisfy (7) and (8), since they are derived from (5).

Conversely, we obtain the first equation of (5) by adding (7) and (8). From equation (7), we obtain

$$\begin{aligned}|\sin(\log(3)y)| &= \left(1-\cos^2(\log(3)y)\right)^{1/2}\\ &= \frac{\left(4\cdot 9^{-x}-(4^{-x}-9^{-x}-1)^2\right)^{1/2}}{2\cdot 3^{-x}}\\ &= \frac{\left((4^{-x}-(3^{-x}-1)^2)((3^{-x}+1)^2-4^{-x})\right)^{1/2}}{2\cdot 3^{-x}}.\end{aligned}$$

Similarly, from (8), we obtain

$$\begin{aligned}|\sin(\log(2)y)| &= \frac{\left((9^{-x}-(2^{-x}-1)^2)((2^{-x}+1)^2-9^{-x})\right)^{1/2}}{2\cdot 2^{-x}}\\ &= \frac{3^{-x}}{2^{-x}}\left|\sin(\log(3)y)\right|.\end{aligned}$$

Thus, the second equation of (6) is satisfied if $\sin(\log(3)y)$ and $\sin(\log(2)y)$ are of opposite sign. $\square$

This gives us an easy method for deciding which intersections of the curves (7) and (8) are actual zeros of $\zeta_3(s)$. We now need a better description of these curves.

Lemma 4.2. *(1) The curve given by equation (7) consists of congruent convex, closed curve segments contained in and tangent at each edge of the disjoint boxes $[-1,r_1]\times[\frac{(2k+1)\pi}{\log(3)}-r_2,\frac{(2k+1)\pi}{\log(3)}+r_2]$ where $k\in\mathbb{Z}$, r_1 is the real root of $1-2^{-x}-3^{-x}=0$ and r_2 is the point where the derivative of $(3/4)^x-3^x-1/3^x$ is 0.*
(2) The curve given by equation (8) is single-valued in x, with maximum x-values obtained at the points $(r_1,\frac{(2k+1)\pi}{\log(2)})$ and minimum x-values at the points $(-1,\frac{2k\pi}{\log(2)})$ for $k\in\mathbb{Z}$. It has positive slope in the half-periods $y\in[\frac{2k\pi}{\log(2)},\frac{(2k+1)\pi}{\log(2)}]$ and negative slope in the half-periods $y\in[\frac{(2k-1)\pi}{\log(2)},\frac{2k\pi}{\log(2)}]$ for $k\in\mathbb{Z}$.

Proof. (1) We start by rewriting equation (7) as

$$\cos(\log(3)y)=\frac{1}{2}\left((3/4)^x-3^x-1/3^x\right)$$

and analyzing the function

$$f(x):=\frac{1}{2}\left((3/4)^x-3^x-1/3^x\right)=\frac{1}{2}(3/4)^x-\cosh((\log(3))x)$$

on the right. For $x > 0$ the term -3^x dominates, while for $x < 0$ the term $-1/3^x$ dominates, so $f(x) \to -\infty$ as $x \to \pm\infty$. Its derivative

$$f'(x) = -\frac{1}{2}\log(4/3)\,(3/4)^x - \sinh((\log(3))x)$$

is the sum of two decreasing functions and so is decreasing, having a single zero at $r_2 = -0.1230\ldots$ with $f(r_2) = -0.4911\ldots$. Thus, $f(x)$ increases up to r_2 and then decreases. The curve is defined only where $\cos(\log(3)y) \le f(r_2)$, i.e., in regions where $\frac{(2k+1)\pi - y_1}{\log(3)} \le y \le \frac{(2k+1)\pi + y_1}{\log(3)}$ with $y_1 = \pi - \arccos(f(r_2)) = 1.0574\ldots < \pi/2$. For each y-value in the interval $\left(\frac{(2k+1)\pi - y_1}{\log(3)}, \frac{(2k+1)\pi + y_1}{\log(3)}\right)$ there are two solutions in x, so that the curve decomposes into disjoint closed paths. The curve achieves its extreme values in x at the points $y = \frac{(2k+1)\pi}{\log(3)}$ where $\cos(\log(3)y) = -1$ giving

$$0 = f(x) + 1 = -\frac{3^x}{2}\left(1 + \frac{1}{2^x} - \frac{1}{3^x}\right)\left(1 - \frac{1}{2^x} - \frac{1}{3^x}\right)$$

A simple analysis shows this has only two real roots, -1 and r_1.

For the curve, we have

$$(\log(3))\sin((\log(3))y)\frac{dy}{dx} = -f'(x).$$

At the extreme values for x, we have $\sin((\log(3))y) = 0$, making the tangents vertical. At the points on the curve with $x = r_2$, we have $f'(x) = 0$, making the tangents horizontal.

The closed paths are symmetric about the lines $y = (2k+1)\pi/\log(3)$. For $y \in [((2k+1)\pi - y_1)/\log(3), (2k+1)\pi/\log(3)]$, we have $\sin((\log(3))y) > 0$, so that the curve is strictly decreasing for $x \in (-1, r_2)$ and strictly increasing for $x \in (r_2, r_1)$.

To determine convexity, we evaluate the second derivative, obtaining

$$\begin{aligned}(\log(3))\sin((\log(3))y)\frac{d^2y}{dx^2} &= (\log(3))^2\frac{(3^x + 3^{-x})}{2} - \log\left(\frac{3}{4}\right)^2\frac{\left(\frac{3}{4}\right)^x}{2} \\ &\quad - (\log(3))^2\cos((\log(3))y)\left(\frac{dy}{dx}\right)^2.\end{aligned}$$

For $y \in [((2k+1)\pi - y_1)/\log(3), (2k+1)\pi/\log(3)]$, we have $\sin((\log(3))y) > 0$, $\cos((\log(3))y) < 0$. Also, for $x > 0, (\log(3))^2 3^x - \log(\frac{3}{4})^2\frac{(\frac{3}{4})^x}{2} > 0$ and for $x < 0, (\log(3))^2 3^{-x} - \log(\frac{3}{4})^2\frac{(\frac{3}{4})^x}{2} > 0$. Thus, $\frac{d^2y}{dx^2} > 0$ and the portions of the curve for these y-values are convex up. Using symmetry, we obtain that the regions enclosed by the closed paths are convex.

(2) Curve (7) is easier to analyze. In this case, let

$$f(x) = \frac{\left(\frac{2}{9}\right)^x - (2^x + 2^{-x})}{2}.$$

Then the curve has the equivalent formulation

$$\cos((\log(2))y) = f(x).$$

For $x < 0$, the term $\left(\frac{2}{9}\right)^x$ dominates, so $\lim_{x\to-\infty} = \infty$. For $x > 0$, the term 2^x dominates, so $\lim_{x\to\infty} = -\infty$. We easily see that $f(x)$ is a strictly decreasing function, since for $x < 0$, both $(2/9)^x - 2^{-x}$ and -2^x are strictly decreasing, while for $x > 0$ both $(2/9)^x$ and $-(2^x + 2^{-x})/2 = -\cosh((\log(2))x)$ are strictly decreasing and $f'(0) = \log(2/9)/2 < 0$. Thus, every y-value corresponds to a single x-value on the curve. Differentiating, we obtain

$$-\sin((\log(2))y)\frac{dy}{dx} = f'(x).$$

Since $f'(x) < 0$ for all x, the sign of $\frac{dy}{dx}$ is the same as the sign of $\sin((\log(2))y)$, giving the result as stated. □

We now have a theorem justifying the pictures shown above.

Theorem 4.3. *(1) The zeros of $\zeta_3(s)$ modulo $2\pi i/\log(3)$ are dense in the curve $1 + 2\cdot 3^{-x}\cos(\log(3)y) + 9^{-x} = 4^{-x}$ over the period $y \in [0, 2\pi/\log(3)]$.*
(2) The zeros of $\zeta_3(s)$ modulo $2\pi i/\log(2)$ are dense in the curve $1+2\cdot 2^{-x}\cos(\log(2)y)+4^{-x} = 9^{-x}$ over the period $y \in [0, 2\pi/\log(2)]$.
(3) The real parts of the zeros of $\zeta_3(s)$ are dense in the interval $[-1, r_1]$.

Proof. (1) This uses the facts that both curves (8) and (7) have the same range of x-values and that the periods $[0, 2\pi/\log(3)]$ and $[0, 2\pi/\log(2)]$ are rationally independent. For a point $(x, y+2k\pi/\log(3))$ on (8) in its first period, consider the point (x, y') on curve (7) in its first period such that $\sin((\log(3))y)$ and $\sin((\log(2))y')$ are of opposite sign. Then we can find $0 < k, l \in \mathbb{Z}$ such that $y + 2k\pi/\log(3)$ and $y' + 2l\pi/\log(2)$ are arbitrarily close. Thus we can find intersection points (x_n, y_n) which are zeros of $\zeta_3(s)$ such that $x_n \to x$ and $y_n \pmod{2\pi/\log(3)} \to y$.
Assertion (2) follows from a like argument while (3) follows easily from either (1) or (2). □

Note: A more general result is given in Corollary 4.8 below.

Theorem 4.4. *All of the zeros of $\zeta_3(s)$ are simple. The are distributed so that each disjoint closed segment of the curve $1 + 2\cdot 3^{-x}\cos(\log(3)y) + 9^{-x} = 4^{-x}$ contains exactly one zero of $\zeta_3(s)$.*

Proof. If a zero of an analytic function

$$F(z) = f(x, y) + ig(x, y)$$

has multiple roots at $z = a + ib$ then

$$F'(a + ib) = \frac{\partial f}{\partial x}(a, b) + i\frac{\partial g}{\partial x}(a, b) = 0.$$

For $\zeta_3(z)$ this translates to the conditions

$$(\log(2))2^{-x}\sin((\log(2))y) + (\log(3))3^{-x}\sin((\log(3))y) = 0 \tag{9}$$

$$(\log(2))2^{-x}\cos((\log(2))y) + (\log(3))3^{-x}\cos((\log(3))y) = 0. \tag{10}$$

From (9) and (6), we obtain $\sin((\log(2))y) = \sin((\log(3))y) = 0$, which is impossible since $\log(2)$ and $\log(3)$ are rationally independent. Therefore, all zeros of $\zeta_3(s)$ must be simple.

For the second claim in the theorem, consider two cases, is illustrated in Figure 7.

Both graphs show the intersection of curve (8) with a single closed segment of curve (7), where curve (8) enters from below with positive slope on the left and with negative slope on the right. We will show in each case that curve (8) has exactly two points of intersection with curve (7) and exits from above with the slope of the same sign as on entry. Thus, $\sin((\log(2)y)$ keeps the same sign at the entry and

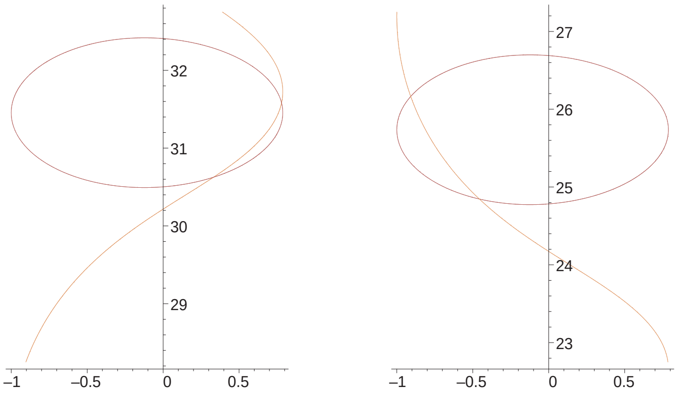

FIGURE 7. Intersections of curves (7) and (8)

exit points while $\sin((\log(3)y)$ changes, so that exactly one of these points is a zero of $\zeta_3(s)$.

Note first that points on each closed segment of curve (7) encompass the full range of x-values of curve (8). Thus, each closed segment of curve (7) must intersect curve (8). Both curves achieve their extreme x-values at y-values which are integer multiples of their half periods, and therefore rationally independent. Thus, these cannot be intersection points for the curves, meaning that curve (8 must intersect at least once with the lower half and at least once with the upper half of each closed segment.

Consider the first case, where the slope of curve (8) is positive, i.e., in one of its first half-periods. We show that wherever this curve intersects with curve (7) in one of its first half-periods, i.e., the lower half of one of the closed segments, the slope of curve (8) is strictly greater that that of curve (7). A second intersection before curve (7) achieves its maximum x-value is then impossible since (8) would then have to cross (7) from left to right and have lower slope.

We rewrite equations (7) and (8) as

$$\begin{aligned} 2\cos(\log(3)y) &= (3/4)^x - 3^x - 1/3^x \\ 2\cos(\log(2)y) &= (2/9)^x - 2^x - 1/2^x \end{aligned}$$

Differentiating, we obtain

$$\begin{aligned} 2(\log(3))\sin((\log(3))y)\frac{dy}{dx} &= (3/4)^x\log(3/4) - (\log(3))(3^x - 1/3^x) \\ 2(\log(2))\sin((\log(2))y)\frac{dy}{dx} &= (2/9)^x\log(2/9) - (\log(2))(2^x - 1/2^x). \end{aligned}$$

At intersection in the first half-periods of both curves we have

$$2^{-x}\sin((\log(2))y) = 3^{-x}\sin((\log(3))y),$$

so that the ratio of the slope of curve (7) to (8) simplifies to

$$R(x) = \frac{\log(2)\,(\log(3)36^x - 9^x\log(3) + 2\log(2)9^x - 4^x\log(3))}{\log(3)\,(\log(2)36^x - 4^x\log(2) + 2\cdot 4^x\log(3) - \log(2)9^x)}.$$

To show that $R(x)$ is increasing over $[-1, r_1]$, we look at its derivative, which simplifies to

$$R'(x) = \frac{4(\log(2))36^x \left((\log(3))^3 4^x - (\log(2))^3 9^x - (\log(3/2))^3\right)}{(\log(3))\left((\log(2))36^x - (\log(2))4^x + 2(\log(3))4^x - (\log(2))9^x\right)^2}.$$

The only factor which changes sign is $f(x) := \left((\log(3))^3 4^x - (\log(2))^3 9^x - (\log(3/2))^3\right)$. We see that $f(x)$ approaches the horizontal asymptote $y = -(\log(3/2))^3$ as $x \to -\infty$ and has a single critical point where $(9/4)^x = (\log(3)/\log(2))^2$ or $x_c = 1.13588\ldots$. Thus, $f(x)$ is increasing on $[-\infty, x_c]$. Since $f(-1) = 0.2278\ldots > 0$, $R'(x) > 0$ on $[-1, r_1]$, and hence, $R(x)$ increases on $[-1, r_1]$. We find $R(-1) = -.4206\ldots$ and $R(r_1) = 0.8684\ldots$, so $|R(x)| < 1$ on $[-1, r_1]$. Since the slope of curve (8) is always positive in this consideration, we have the slope of curve (8) is strictly greater than that of curve (7) at any point of intersection, as claimed.

In the second case shown in Figure 7, the slope of curve (7) is negative. Where it intersects with curve (8) in one of its first half-periods we have

$$2^{-x}\sin((\log(2))y) = 3^{-x}\sin((\log(3))y),$$

For the ratio of slopes, $R(x)$, is this case, we again have $|R(x)| < 1$, meaning that the slope of curve (8) is strictly less than that of curve (7). Thus, there is only one intersection in this first half-periods of curve (8).

By the symmetry of the cosine functions for y-values, curve (7) has a single intersection in each case with the second half-periods of curve (8). These must be in the same half-periods of curve (7). For the two intersections in each case, the sign of $\sin((\log(2))y)$ remains the same while that of $\sin((\log(3))y)$ changes. The theorem follows then, from Lemma 4.1. □

The occurrance of periods is expressed more generally in the following theorem.

Theorem 4.5. *Let $\zeta_{N,p,q}(s)$ be an exponential sum composed of terms c_n/n^s, where c_n is constant and n is divisible only by the primes p and q. Then the zeros of $\zeta_{N,p,q}(s)$ reduced modulo*

$$\left|\frac{2\pi i}{m_1\log(p) + m_2\log(q)}\right|$$

lie on a period of a curve, where $(m_1, m_2) = (0,1)$ or $(1,0)$ or $m_1, m_2 \in \mathbb{Z}$ with $gcd(m_1, m_2) = 1$.

Proof. Let c_n/n^s be one of the terms of $\zeta_{N,p,q}(s)$ with $n = p^{e_1}q^{e_2}$. Then

$$\begin{aligned} 1/n^s &= p^{-e_1x}q^{-e_2x}\left(\cos(\log(p^{e_1}q^{e_2})) + i\sin(\log(p^{e_1}q^{e_2}))\right) \\ &= p^{-e_1x}q^{-e_2x}\left(\cos\left((e_1\log(p) + e_2\log(q))y\right) + i\sin\left((e_1\log(p) + e_2\log(q))y\right)\right). \end{aligned}$$

Using trigonometric summation formulas, we can express the real and imaginary parts of $\zeta_{N,p,q}(s)$ as polynomials in p^{-x}, q^{-x} and the sine and cosine functions of $(\log(p))y$ and $(\log(q))y$.

Let $\gcd(m_1, m_2) = 1$. Then we can find $l_1, l_2 \in \mathbb{Z}$ with $l_1m_2 - l_2m_1 = 1$. Solving the linear system

$$\begin{aligned} (\log(q))y &= l_1A + m_1B \\ (\log(p))y &= -l_2A - m_2B \end{aligned}$$

in A and B, we obtain.

$$\begin{aligned} A &= (m_1(\log(p)) + m_2(\log(q)))\, y \\ B &= -\,(l_1(\log(p)) + l_2(\log(q)))\, y. \end{aligned}$$

Using trigonometric summation formulas again , we can express the real and imaginary parts of $\zeta_{N,p,q}(s)$ as polynomials in p^{-x}, q^{-x} and the sine and cosine functions of A and B. Now we can use Groebner basis reduction to eliminate $\cos(B)$ and $\sin(B)$ from the equations $\Re\left(\zeta_{N,p,q}(s)\right) = 0$, $\Im\left(\zeta_{N,p,q}(s)\right) = 0$, and $\cos^2(B) + \sin^2(B) = 0$, obtaining a single equation $F(x, \cos(A), \sin(A)) = 0$. The zeros of $\zeta_{N,p,q}(s)$ lie on this curve, which has period $|2\pi i/(m_1(\log(p)) + m_2(\log(q)))|$ in y. $\square$

Some examples of the application of this theorem are given below:

(1) On the left in Figure 8 we see zeros of $\zeta_3(s)$ reduced modulo $3\left(\frac{2\pi i}{\log(2^6 3^7)}\right)$. We

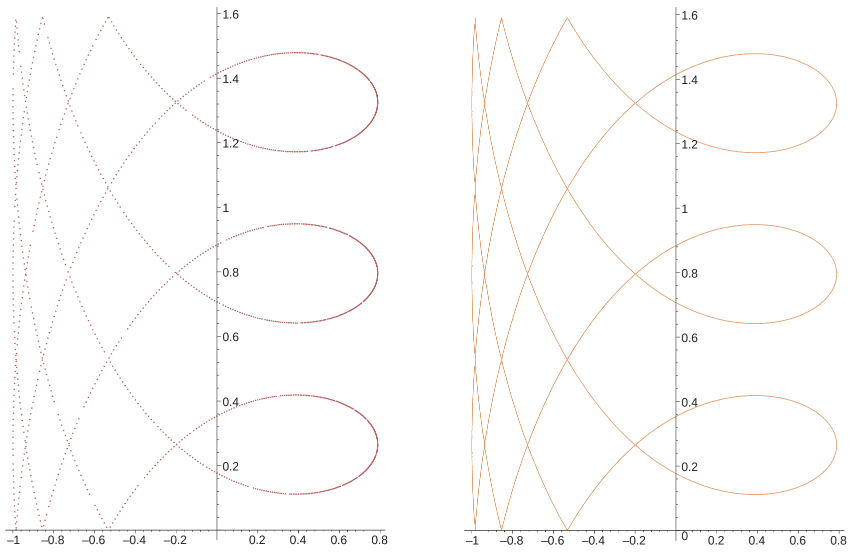

FIGURE 8. Zeros of $\zeta_3(s)$ reduced modulo $\frac{6\pi i}{\log(2^6 3^7)}$ and derived curve

apply Theorem 4.5 with $p = 2, q = 3, m_1 = 6, m_2 = 7$, deriving $l_1 = l_2 = 1$. Then, we have

$$\begin{aligned} (\log(2))y &= -A - 7B \\ (\log(3))y &= A + 6B \end{aligned}$$

where $A = 6\log(2) + 7\log(3)$, $B = -\log(2) - \log(3)$. Carrying out the algebraic reduction to remove the trigonometric terms in B, we obtain $2\cos(A)\cdot\ 139968^{-x} + 4096^{-x} - 6\cdot\ 43046721^{-x} - 153\cdot\ 688747536^{-x} + 78\cdot\ 502096953744^{-x} + 1287\cdot\ 3869835264^{-x} - 405\cdot\ 2754990144^{-x} - 825\cdot\ 11019960576^{-x} + 21\cdot\ 65536^{-x} - 21\cdot\ 4194304^{-x} - 1287\cdot\ 44079842304^{-x} + 252\cdot\ 241864704^{-x} - 253\cdot\ 55788550416^{-x} - 1254\cdot\ 967458816^{-x} - 78\cdot\ 339738624^{-x} - 825\cdot\ 191102976^{-x} + 210\cdot\ 1224440064^{-x} + 605\cdot\ 24794911296^{-x} + 715\cdot\ 99179645184^{-x} + 462\cdot\ 4897760256^{-x} + 462\cdot\ 2176782336^{-x} + 94\cdot\ 1549681956^{-x} -$

$1716 \cdot 8707129344^{-x} + 1320 \cdot 429981696^{-x} - 715 \cdot 1719926784^{-x} + 59 \cdot 125524238436^{-x} + 126 \cdot 26873856^{-x} + 7 \cdot 19131876^{-x} + 125 \cdot 589824^{-x} - 41 \cdot 172186884^{-x} - 71 \cdot 37748736^{-x} - 286 \cdot 223154201664^{-x} - 20 \cdot 3486784401^{-x} + 1716 \cdot 19591041024^{-x} - 750 \cdot 107495424^{-x} - 43 \cdot 147456^{-x} + 160 \cdot 9437184^{-x} + 855 \cdot 47775744^{-x} - 7 \cdot 16384^{-x} - 67108864^{-x} - 35 \cdot 262144^{-x} + 13 \cdot 150994944^{-x} + 319 \cdot 84934656^{-x} + 35 \cdot 1048576^{-x} + 6 \cdot 36864^{-x} + 2541865828329^{-x} - 6 \cdot 282429536481^{-x} + 7 \cdot 16777216^{-x} - 372 \cdot 11943936^{-x} - 13 \cdot 1129718145924^{-x} + 84 \cdot 306110016^{-x} - 190 \cdot 2359296^{-x} + 396 \cdot 5308416^{-x} - 148 \cdot 1327104^{-x} - 106 \cdot 13947137604^{-x} - 510 \cdot 21233664^{-x} + 21 \cdot 331776^{-x} + 4782969^{-x} + 28 \cdot 76527504^{-x} + 15 \cdot 387420489^{-x} + 300 \cdot 6198727824^{-x} + 286 \cdot 764411904^{-x} + 15 \cdot 31381059609^{-x} + 56 \cdot 2985984^{-x} = 0$, giving the plot on the right. This expression has 64 non-zero terms, with degrees 1 in $\cos(\log(2^6 3^7)y)$ and 26 in each 2^{-x} and 3^{-x} for comparison with data for other curves given below.

(2) Figure 9 shows zeros of η_4 reduced modulo $\frac{2\pi i}{\log(2)}$ and modulo $\frac{2\pi}{\log(3)}$. For the

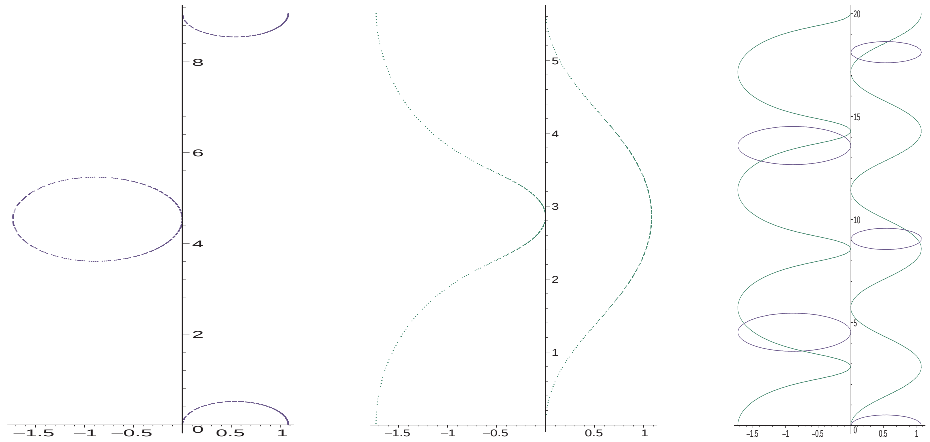

FIGURE 9. 500 zeros of $\eta_4(s)$ reduced modulo $\frac{2\pi i}{\log(2)}$, modulo $\frac{2\pi i}{\log(3)}$ and derived curves

first diagram, the zeros lie on the curve given by

$$4^{-x+1}\cos(\log(2)y)^2 + 2\left(2^{-x} - 8^{-x}\right)\cos(\log(2)y) - 3 \cdot 4^{-x} + 9^{-x} - 16^{-x} - 1 = 0.$$

This splits into linear factors in $\cos(\log(2))$, one giving the loops to the left of $y = 0$ and the other giving toe loops to the right. For the second diagram, the curves are given by

$$\begin{aligned}&4\cos(\log(3)y)^2 \cdot 9^{-x} + \left(4 \cdot 27^{-x} + 4 \cdot 3^{-x} - 6 \cdot 48^{-x} - 2 \cdot 12^{-x}\right)\cos(\log(3)y)\\ &+256^{-x} - 64^{-x} - 2 \cdot 144^{-x} - 36^{-x} + 81^{-x} - 4^{-x} + 2 \cdot 9^{-x} - 4 \cdot 16^{-x} + 1 = 0.\end{aligned}$$

This again splits, giving curves to the left and right of $y = 0$. The third plot shows several periods of these curves plotted together.

(3) We consider the function

$$\begin{aligned}\pi_{24}(s) \quad &:= \quad 1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{6^s}+\frac{1}{8^s}+\frac{1}{9^s}\\ &\qquad +\frac{1}{12^s}+\frac{1}{16^s}+\frac{1}{18^s}+\frac{1}{24^s}\end{aligned}$$

Here the terms involve only the primes 2 and 3, so by Theorem 4.5 the zeros reduced modulo $\frac{2\pi i}{\log(24)}$ will lie on a curve in the plane. Using the algebraic reduction techniques outlined we may derive an explicit formula for this curve, the data on this is contained in the following table. Figure 10 depicts the curve. Notice that

	no. of terms	$\deg\left(\cos(\log(24)\tilde{t})\right)$	$\deg(2^{-\sigma})$	$\deg(3^{-\sigma})$
π_{24}	1765	8	80	40

TABLE 1. Data of the curve of the reduced zeros of π_{24}

the picture is turned -90 degrees, to obtain a more dramatic effect.

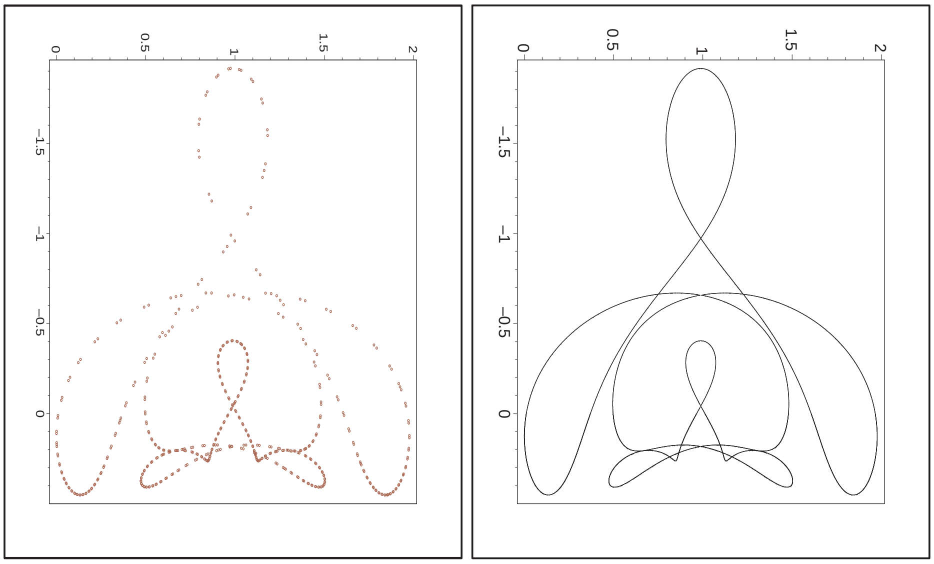

FIGURE 10. 506 zeros of π_{24} reduced modulo $\frac{2\pi i}{\log(24)}$ and derived curve. (Alexa's penguin)

4.2. **Multiple parameter curves:** We now examine the patterns which appear when the imaginary parts of the zeros of an exponential sum involving three or more primes are reduced via various periods. We start by reconsidering Figure 1 from the introduction where zeros of $\zeta_5(s)$ are reduced modulo $2\pi i/\log(5)$. There no longer appears to be a simple curve. Since the number of zeros is countable, they cannot fill a region. However, there are areas where they appear to become dense. There are also shadings and outlines for curves which ask for explanation.

The zeros of $\zeta(s)$ are the intersection points of the curves

$$1+\frac{\cos(\log(2)y)}{2^x}+\frac{\cos(\log(3)y)}{3^x}+\frac{\cos(2\log(2)y)}{2^{2x}}+\frac{\cos(\log(5)y)}{5^x} = 0 \tag{11}$$

$$\frac{\sin(\log(2)y)}{2^x}+\frac{\sin(\log(3)y)}{3^x}+\frac{\sin(2\log(2)y)}{2^{2x}}+\frac{\sin(\log(5)y)}{5^x} = 0. \tag{12}$$

We can eliminate the terms in $\log(3)y$, say, but we are still left with trigonometric terms in both $\log(2)y$ and $\log(5)y$, so the function we obtain no longer has a simple period in y as before. We do note, however, that for values $y_1 \pmod{2\pi/\log(2)}, y_2 \pmod{2\pi/\log(3)}, y_3 \pmod{2\pi/\log(5)}$, we can find y so that the values $y-y_1 \pmod{2\pi/\log(2)}, y-y_2 \pmod{2\pi/\log(3)}, y-y_3 \pmod{2\pi/\log(5)}$ are arbitrarily small. This leads us to consider $\log(2)y, \log(3)y$ and $\log(5)y$ as independent variables $\log(2)y_1, \log(3)y_2$ and $\log(5)y_3$. The function

$$F(x,y_1,y_2,y_3) = 1+2^{x+iy_1}+3^{x+iy_2}+2^{2(x+iy_1)}+5^{x+iy_3} \tag{13}$$

maps $\mathbb{R}^4$ into $\mathbb{C}$. The zeros of F are not discrete, but lie on smooth surfaces. F is periodic the the variables y_1, y_2, y_3 with periods $\pi_1 = 2\pi/\log(2), \pi_2 = 2\pi/\log(3), \pi_3 = 2\pi/\log(5)$ respectively. Thus, the zeros of F in $\mathbb{R}\times\prod_{j=1}^{3}[0,\pi_j]$ may be taken as representatives of all the zeros of F. The partial diagonal map $I : x+iy \to (x,y,y,y)$ takes $\mathbb{C}$ into $\mathbb{R}^4$. We would like to investigate the relationship between the zeros of $\zeta_5(s)$ via the map I and the zeros of F. We approach this more generally in the following theorem.

Theorem 4.6. *Let $F(x,y_1,y_2,\ldots,y_k) : \mathbb{R}^{k+1} \to \mathbb{C}$ be continuously differentiable and periodic in the variables $y_1,y_2,\ldots,y_k$ with periods $\pi_1,\pi_2,\ldots,\pi_k$ which are rationally independent. Suppose that the function $F(x,y,\ldots,y)$ obtained by restriction to the partial diagonal $y_1 = y_2 = \ldots = y_k = y$ defines a complex analytic function $f(x+iy)$ and that $\{(x,y_1,y_2,\ldots,y_k) : \frac{\partial F}{\partial x} \neq 0\}$ is dense in $\mathbb{R}^{k+1}$. Let $I : \mathbb{C}\to\mathbb{R}^{k+1}$ be the injection $x+iy \to (x,y,y,\ldots,y)$ and $R : \mathbb{R}^{k+1} \to \mathbb{R}\times\prod_{j=1}^{k}[0,\pi_j]$ be the reduction map taking y_j to its equivalent modulus in the period $[0,\pi_j]$. The the zeros of f under the composition $R\circ I$ are dense in the zeros of F in $\mathbb{R}\times\prod_{j=1}^{k}[0,\pi_j]$.*

Proof. Let $X_z = (x,y_1,y_2,\ldots,y_k) \in \mathbb{R}\times\prod_{j=1}^{k}[0,\pi_j]$ be a zero of F. We wish to show that for $\varepsilon > 0$ we can find a zero z of f and $\overline{X_z} \in R^{-1}(X_z)$ such that $|I(z)-\overline{X_z}| < \varepsilon$. By the hypothesis, we may restrict ourselves to X_z for which $a = \frac{\partial F}{\partial x} \neq 0$. Certainly, we may find z and $\overline{X} \in R^{-1}(X_z)$ such that $I(z)$ and $\overline{X}$ are as close as we may like. The problem is to find such a z with $f(z) = 0$. We start by finding z_0 and $\overline{X_z}$ close enough and then show that the iterates obtained using Newton's method

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}$$

converge to the desired z.

First, by the continuity of the derivatives, we can find $\delta_1 < \varepsilon$ such that

$$|X - X_z| < \delta_1 \quad\Rightarrow\quad \left|\frac{\partial F}{\partial x}(X) - a\right| < \frac{\min\{|a|,1\}}{8(k+1)}.$$

Further, by continuity, we can find $\delta_2 < \delta_1$ such that

$$|X - X_z| < \delta_2 \quad\Rightarrow\quad |F(X)| < \frac{\delta_1|a|}{8(k+1)}.$$

We now find $z_0 \in \mathbb{C}$ such that $|I(z_0) - \overline{X_z}| < \delta_2/2$ for some $\overline{X_z} \in R^{-1}(X)$. We note the following
(1) $f'(x+iy) = \frac{\partial f}{\partial x}(x+iy) = \frac{\partial F}{\partial x}(I(x+iy))$.
(2) If $|I(z') - \overline{X_z}| < \delta_1$ and $|I(z'') - \overline{X_z}| < \delta_1$ then $\left|\frac{f(z'')-f(z')}{z''-z'}\right| < \frac{\min\{|a|,1\}}{8(k+1)}$. Otherwise, for some point z''' on the line between z' and z'', we would have $|f'(z''')| \geq \frac{\min\{|a|,1\}}{8(k+1)}$. However, since $|I(z''') - \overline{X_z}| < \delta_1$, this contradicts our choice for δ_1.
(3) $|f'(z) - a| < |a|/8(k+1) \Rightarrow |f'(z)| \geq (8k+7)|a|/8(k+1)$
(4) In $\mathbb{C}$, $|z' - z''| < \delta/(k+1) \quad \Rightarrow \quad |I(z') - I(z'')| < \delta$ in $\mathbb{R}^{k+1}$.

We now examine the Newton's method iterates. By the choice of δ_2 we have

$$\begin{aligned}
|z_1 - z_0| &= \left|\frac{f(z_0)}{f'(z_0)}\right| = \left|\frac{f(z_0)}{a} + f(z_0)\left(\frac{1}{f'(z_0)} - \frac{1}{a}\right)\right| \\
&\leq \left|\frac{F\circ I(z_0)}{a}\right| + \left|\frac{F\circ I(z_0)}{a}\right|\left|\frac{F\circ I(z_0) - a}{F\circ I(z_0)}\right| \\
&= \frac{\delta_1}{8(k+1)} + \frac{\delta_1}{8(k+1)}\frac{|a|}{8(k+1)}\frac{8(k+1)}{|a|(8k+7)} \\
&< \frac{\delta_1}{4(k+1)}
\end{aligned}$$

This implies $|I(z_1) - I(z_0)| < \delta_1/4$ and ensures that $|I(z_1) - \overline{X_z}| < 3\delta_1/4$, so we still have control over the size of the derivative.

We assume inductively that $|I(z_{n+1}) - \overline{X_z}| < \delta_1$ and $|I(z_n) - \overline{X_z}| < \delta_1$, which is true for $n = 0$. Then, we have

$$\begin{aligned}
\left|\frac{f(z_{n+1})}{z_{n+1} - z_n}\right| &= \left|\frac{f(z_{n+1} - f(z_n)}{z_{n+1} - z_n} - f'(z_n)\right| \\
&\leq \left|\frac{f(z_{n+1} - f(z_n)}{z_{n+1} - z_n} - a\right| + |a - f'(z_n)| \\
&< \frac{\min\{|a|,1\}}{4(k+1)}.
\end{aligned}$$

Then we have

$$\begin{aligned}
|z_{n+2} - z_{n+1}| &= \left|\frac{f(z_{n+1})}{f'(z_{n+1})}\right| \\
&< \frac{\min\{|a|,1\}}{4(k+1)}|z_{n+1} - z_n|\frac{8(k+1)}{|a|(8k+7)} \\
&< \frac{|z_{n+1} - z_n|}{2(k+1)}.
\end{aligned}$$

This gives $|I(z_{n+2}) - I(z_{n+1})| < \frac{1}{2}|I(z_{n+2}) - I(z_{n+1})|$ so we always have $|I(z_n) - \overline{X_z}| < \delta_1$ as our Newton's method proceeds. The sequence z_n converges to some $z \in \mathbb{C}$ with $f(z) = F(I(z)) = 0$. Also, $|I(z) - \overline{X_z}| \leq \delta_1 < \varepsilon$, as required. $\square$

We apply this theorem along with Theorem 3.1 to establish a lower bound for the real parts of zeros of $\zeta_N(s)$ for N prime.

Theorem 4.7. *For N prime we have*

$$\inf\{\sigma : \zeta_N(s) = 0\} = \beta_{_N}$$

where $\beta_{_N}$ is the negative root of $\zeta_{_{N-1}}(\beta_{_N}) = N^{-\beta_N}$.

Proof. Suppose that p_j is the jth prime with $p_k = N$. Substituting $\log(p_j)t = y_j$ in the equations $\Re((\zeta_N(s)) = 0$ and $Im((\zeta_N(s)) = 0$, we obtain

$$1 + 2^{-x}\cos(y_1) + 3^{-x}\cos(y_2) + \ldots + N^{-x}\cos(y_k) = 0$$
$$1 + 2^{-x}\sin(y_1) + 3^{-x}\sin(y_2) + \ldots + N^{-x}\sin(y_k) = 0,$$

we see that the substutions $y_j = 0$ for $1 \le j \le N-1$ and $y_k = \pi$ cause the second equation to vanish and transform the first to

$$1 + 2^{-x} + 3^{-x} + \ldots + (N-1)^{-x} - N^{-x} = 0,$$

for which the negative root is $\beta_{_N}$. Applying Theorem 4.6, we can find roots of $\zeta_N(\sigma + it)$ for which σ is arbitrarily close to $\beta_{_N}$. Applying Theorem 3.1, the result follows. □

To relate Theorem 4.6 back to our diagrams, we use the projection $P : \mathbb{R}^{k+1} \to \mathbb{C}$ defined by $P(x, y_1, \ldots, y_k) = x + iy_k$. The next corollary follows directly.

Corollary 4.8. *Let F, f be as in the theorem and let V be the zeros of F in $\mathbb{R} \times \prod_{j=1}^{k}[0, \pi_j]$. Then the zeros of f reduced modulo $i\pi_k$ into the period $0 \le y < \pi_k$ are dense in $P(V)$.*

We return now to Figure 1. The zeros of $\zeta_5(s)$ reduced modulo $2\pi i/\log(5)$ are dense in the P-image of the zeros of $F(x, y_1, y_2, y_3)$, as defined in (13), in $\mathbb{R} \times \prod_{j=1}^{3}[0, \pi_j]$. We can obtain equations for bounding curves by examining the extreme x-values of zeros of F. Where F_r and F_i are the real and imaginary parts of F, these zeros satisfy

$$(14)\ F_r = 1 + \frac{\cos(\log(2)y_1)}{2^x} + \frac{\cos(\log(3)y_2)}{3^x} + \frac{\cos(2\log(2)y_2)}{2^{2x}} + \frac{\cos(\log(5)y_3)}{5^x} = 0$$

$$(15)\ F_i = \frac{\sin(\log(2)y_1)}{2^x} + \frac{\sin(\log(3)y_2)}{3^x} + \frac{\sin(2\log(2)y_1)}{2^{2x}} + \frac{\sin(\log(5)y_3)}{5^x} = 0.$$

We regard (15) as defining y_2 as a function of x, y_1, y_2, with F_r then giving x as a function of the independent variables y_1 and y_3. Taking partial derivatives with respect to y_1, we obtain

$$\begin{aligned} \frac{\partial F_r}{\partial x}\frac{\partial x}{\partial y_1} + \frac{\partial F_r}{\partial y_1} + \frac{\partial F_r}{\partial y_2}\frac{\partial y_2}{\partial y_1} &= 0 \\ \frac{\partial F_i}{\partial x}\frac{\partial x}{\partial y_1} + \frac{\partial F_i}{\partial y_1} + \frac{\partial F_i}{\partial y_2}\frac{\partial y_2}{\partial y_1} &= 0. \end{aligned}$$

To obtain the extreme x values we set $\frac{\partial x}{\partial y_1} = 0$. Eliminating $\frac{\partial y_2}{\partial y_1}$ we obtain the condition

$$\frac{\partial F_r}{\partial y_1}\frac{\partial F_i}{\partial y_2} - \frac{\partial F_r}{\partial y_2}\frac{\partial F_i}{\partial y_1} = 0.$$

Note that regarding y_1 as the dependent variable and y_2 independent leads to the same condition.

For (13), we obtain the equation

$$\frac{\sin(\log(3)y_2)}{3^x}\left(\frac{\cos(\log(2)y_1)}{2^x}+\frac{2\cos(2\log(2)y_1)}{2^{2x}}\right)- \tag{16}$$

$$\left(\frac{\sin(\log(2)y_1)}{2^x}+\frac{2\sin(2\log(2)y_1)}{2^{2x}}\right)\frac{\cos(\log(3)y_2)}{3^x}=0. \tag{17}$$

We now have the three equations (14), (15) and (16) to describe the bounding curve. Using algebraic reduction we can eliminate terms in $\log(2)y_1$ and $\log(3)y_2$ to obtain a single equation involving x and $\log(5)y_3$ having period $2\pi i/\log(5)$. In Figure 11 we see the comparison between reduced zeros of $\zeta_5(s)$ and a plot of the image the curve given by this equation under the projection P.

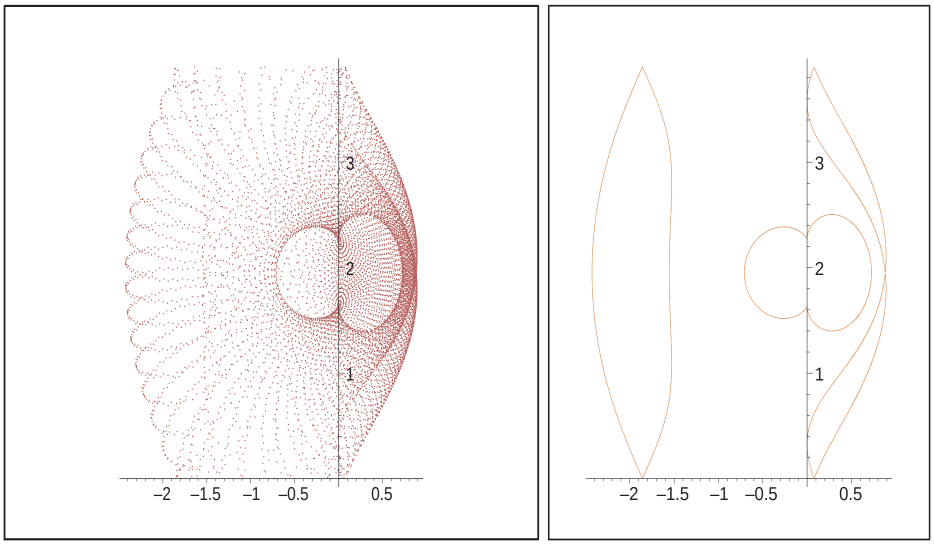

FIGURE 11. Zeros of $\zeta_5(s)$ reduced modulo $\frac{2\pi i}{\log(5)}$ with bounding curve

Applying the same technique, we can derive the equation for bounding curve for the zeros of $\eta_5(s)$ reduced modulo $2\pi i/\log(5)$. These are illustrated in Figure 12. Of interest here is the crescent moon shaped area which appears to exclude zeros. Data on the equations derived for these bounding curves is given in Table 2.

	no. of terms	deg $(\cos(\log(5)\tilde{t}))$	deg$(2^{-\sigma})$	deg$(3^{-\sigma})$	deg$(5^{-\sigma})$
ζ_5	183	5	16	8	10
η_5	185	5	16	8	10

TABLE 2. Data of the bounding curves of ζ_5 and η_5

The algebraic reduction entailed is already fairly involved. For $\zeta_6(s)$ and $\eta_6(s)$ the data on the equations for the bounding curves is given in Table 3. These evaluations were done using Maple 8, and seem to approach the limit of what current algebraic packages are able to practically handle.

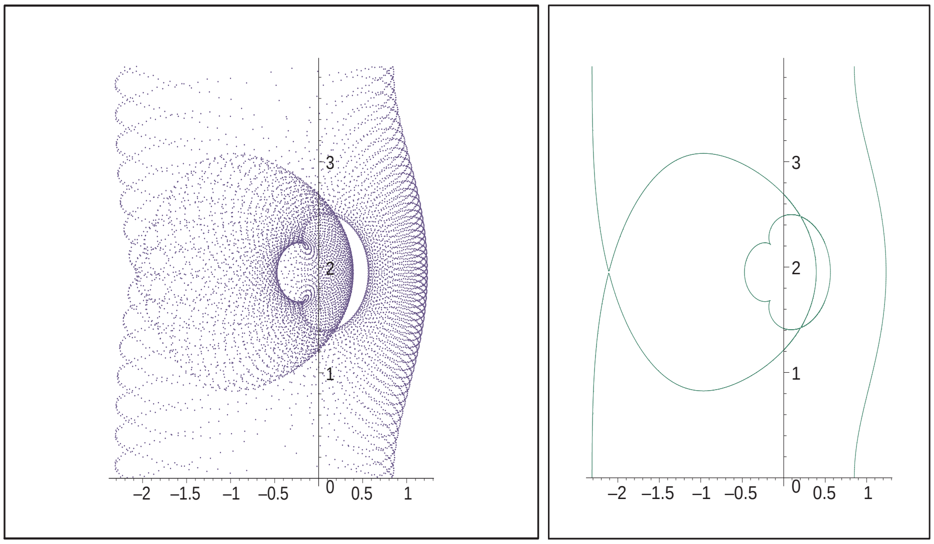

FIGURE 12. 10,000 zeros of $\eta_5(s)$ reduced modulo $\frac{2\pi i}{\log(5)}$ with bounding curve.

	no. of terms	$\deg\left(\cos(\log(6)\tilde{t})\right)$	$\deg(2^{-\sigma})$	$\deg(3^{-\sigma})$	$\deg(5^{-\sigma})$
ζ_6	1197	7	30	18	12
η_6	1202	7	30	18	12

TABLE 3. Data of the bounding curves of ζ_5 and η_6

For these examples, what are probably the more obvious periods, i.e., $2\pi i/\log(N)$ were chosen for reducing zeros. As in the case with two prime exponential sums, we can choose a period $2\pi i/L$, where L is a rational linear combination of $\log(p_j)$'s for primes p_j appearing in the sum. The finer structures of these diagrams are more apparent in the sums involving three primes, see Figures 15 and 16 below. With more primes, the diagrams for some periods often seem fairly amorphous. However, as we see in Figure 17, the choice of an appropriate period may still give a diagram with interesting separations of points.

We thank Nils Bruin of the University of Sydney for conducting the extended calculation to derive the equation of the curve shown in Figure 10 using the computer algebra system Magma [BCP97].

REFERENCES

[BCP97] Wieb Bosma, John Cannon, and Catherine Playoust. The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265, 1997. Computational algebra and number theory (London, 1993).

[Con89] J. B. Conrey. More than two fifths of the zeros of the Riemann zeta function are on the critical line. *J. Reine Angew. Math.*, 399:1–26, 1989.

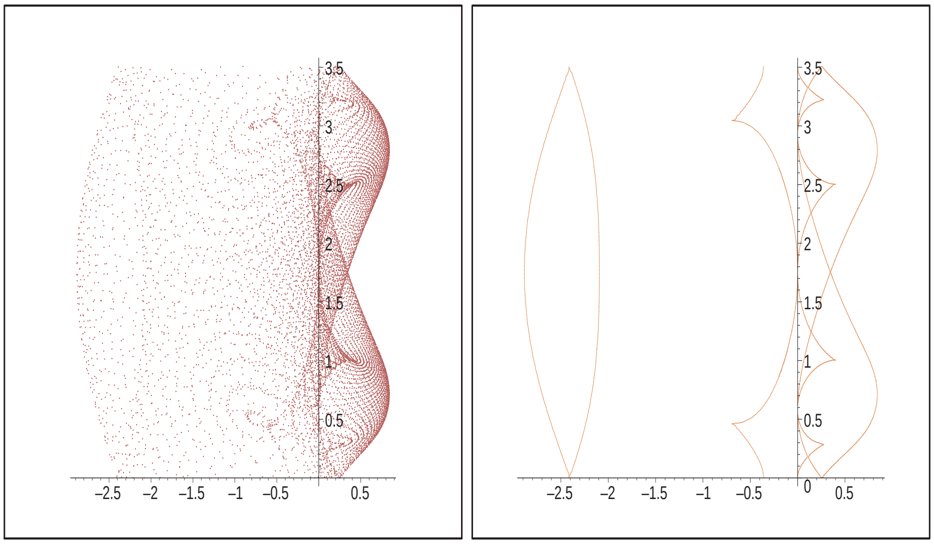

FIGURE 13. 12,000 zeros of $\zeta_6(s)$ reduced modulo $\frac{2\pi i}{\log(6)}$ with bounding curve

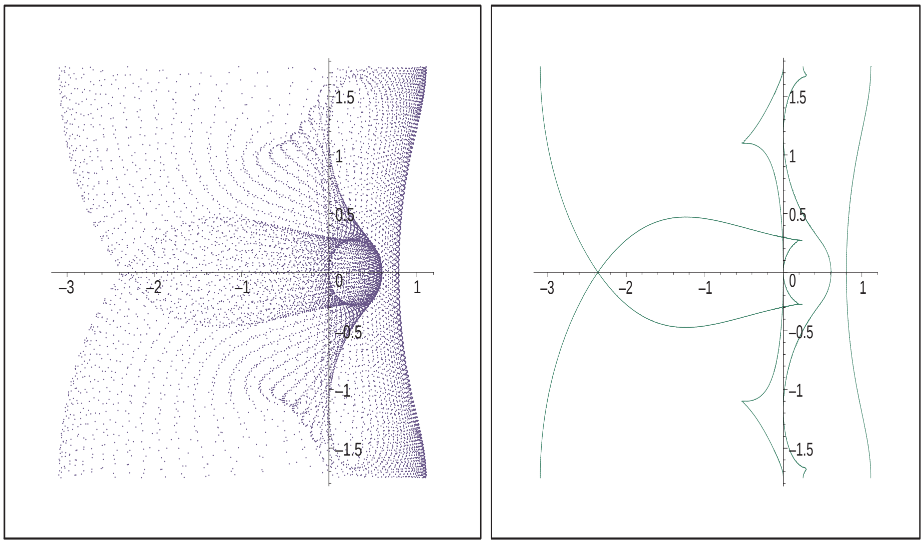

FIGURE 14. 12,000 zeros of $\eta_6(s)$ reduced modulo $\frac{2\pi i}{\log(6)}$ and bounding curve

[Con03] J. Brian Conrey. The Riemann hypothesis. *Notices Amer. Math. Soc.*, 50(3):341–353, 2003.

[Edw01] H. M. Edwards. *Riemann's zeta function*. Dover Publications Inc., Mineola, NY, 2001. Reprint of the 1974 original [Academic Press, New York; MR **57** #5922].

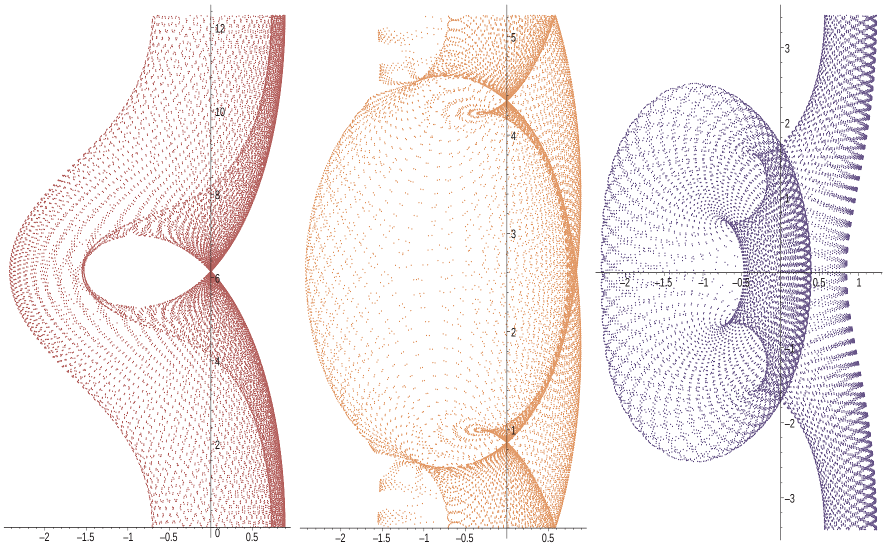

FIGURE 15. 20,000 zeros of $\zeta_5(s)$ reduced modulo (1) $\frac{2\pi i}{\log(5/3)}$ and (2) $\frac{2\pi i}{\log(10/3)}$. 20,000 zeros of $\eta_5(s)$ reduced modulo (3) $\frac{2\pi i}{\log(5/2)}$

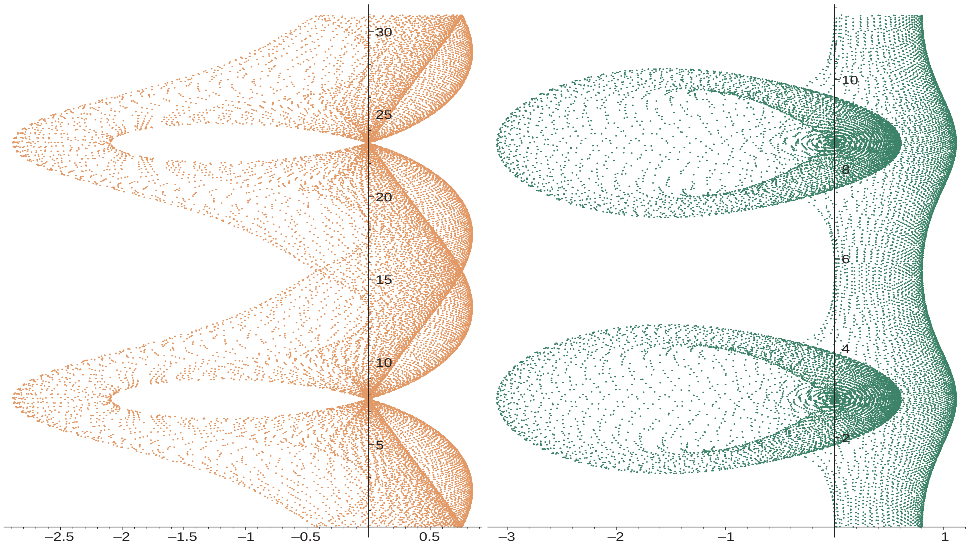

FIGURE 16. 20,000 zeros of $\zeta_6(s)$ and $\eta_6(s)$ reduced modulo $\frac{4\pi i}{\log(3/2)}$ and $\frac{4\pi i}{\log(3)}$, respectively.

[Jen18] R. Jentzsch. Untersuchungen zur Theorie der Folgen analytischer Functionen. *Acta Math.*, 41:219–251, 1918.

[Lan31] R. E. Langer. On the zeros of exponential sums and integrals. *Bull. Amer. Math. Soc.*, 37:213–239, 1931.

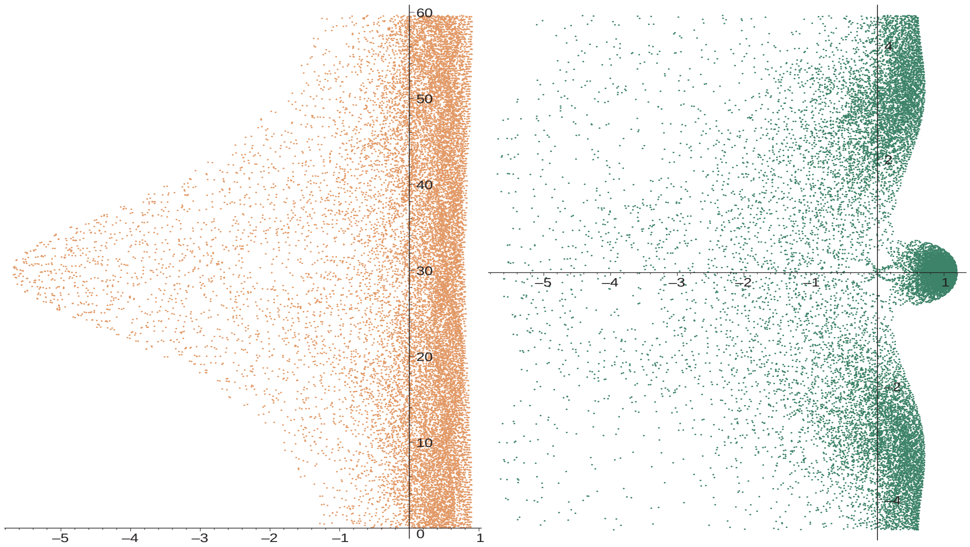

FIGURE 17. 20,000 zeros of $\zeta_{10}(s)$ and $\eta_{10}(s)$ reduced modulo $\frac{2\pi i}{\log(10/9)}$ and $\frac{2\pi i}{\log(2)}$ respectively.

[Mon83] H. L. Montgomery. Zeros of approximations to the zeta function. In *Studies in pure mathematics*, pages 497–506. Birkhäuser, Basel, 1983.

[Pat88] S. J. Patterson. *An introduction to the theory of the Riemann zeta-function*. Cambridge University Press, Cambridge, 1988.

[Spi66] Robert Spira. Zeros of sections of the zeta function. I. *Math. Comp.*, 20:542–550, 1966.

[Tit86] E. C. Titchmarsh. *The theory of the Riemann zeta-function*. The Clarendon Press Oxford University Press, New York, second edition, 1986. Edited and with a preface by D. R. Heath-Brown.

[Tur48] Paul Turán. On some approximative Dirichlet-polynomials in the theory of the zeta-function of Riemann. *Danske Vid. Selsk. Mat.-Fys. Medd.*, 24(17):36, 1948.

[Tur59] P. Turán. Nachtrag zu meiner Abhandlung "On some approximative Dirichlet polynomials in the theory of zeta-function of Riemann". *Acta Math. Acad. Sci. Hungar.*, 10:277–298 (unbound insert), 1959.

[Wil17] Charles E. Wilder. Expansion problems of ordinary linear differential equations with auxiliary conditions at more than two points. *Trans. Amer. Math. Soc.*, 18(4):415–442, 1917.

DEPARTMENT OF MATHEMATICS, SIMON FRASER UNIVERSITY, BURNABY, BRITISH COLUMBIA, CANADA V5A 1S6