The exponential growth of lattice paths.

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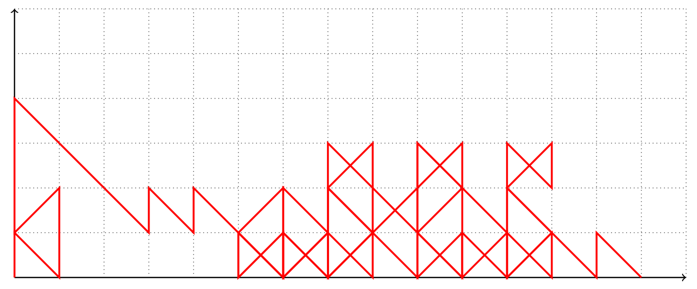
Introduction

A lattice path model is a combinatorial class defined by a **region** and a **direction set** $\mathfrak{S} \subseteq \{0,1,-1\} \times \{0,1,-1\}$.

eg. $\mathcal{Q}_{\curlywedge}$ = the set of walks in the first quadrant with steps from $\mathfrak{S} = \{(0,1),(1,-1),(-1,-1)\}$.

An element of $\mathcal{Q}_{\curlywedge}$ with 100 steps:



Let $q_{\mathfrak{S}}(n)$ be the # of walks of length n in the first quadrant. Typically

$$q_{\mathfrak{S}}(n) \sim \kappa \beta^n n^\alpha, \quad \kappa \in \mathbb{R}^+, \alpha \leq 0.$$

Goal: Given model $\mathcal{Q}_{\mathfrak{S}}$, find $\beta_{\mathfrak{S}}$.

This is the exponential growth factor. We write $q_{\mathfrak{S}} \bowtie \beta$.

Motivation

- In statistical mechanical applications, the exponential growth is the **limiting free energy**, linked to the entropy of the system.
- Although we can estimate β with series computations, we prefer an approach that is **direct, systematic and combinatorial**.
- Experimentally we see that the **drift**

  eg. $\delta(\curlywedge) = \downarrow$

  is key. We would like to explain the link in detail.

History

- **Full plane:** Trivial, factor is always $\beta_{\mathfrak{S}} = |\mathfrak{S}|$.
- **Half plane:** Drift dependent, fully explicit results based on singularity analysis of Banderier and Flajolet [1].
- **Quarter plane:** Experimental results from series analysis are known [2], as well as several enumerative strategies. A few sporadic cases are solved [3].

Conjectured values of Bostan and Kauers

For 23 models with vertical drift, Bostan and Kauers found the following asymptotic expressions in [2].

$\mathfrak{S}$	δ	κ	α	$\beta_{\mathfrak{S}}$	$\mathfrak{S}$	δ	κ	α	$\beta_{\mathfrak{S}}$
+	·	$\frac{4}{\pi}$	-1	4	×	·	$\frac{2}{\pi}$	-1	4
⋇	·	$\frac{\sqrt{6}}{\pi}$	-1	6	✳	·	$\frac{8}{\pi}$	-1	8
Y	↑	$\frac{\sqrt{3}}{\sqrt{\pi}}$	$-\frac{1}{2}$	3	Y	↑	$\frac{\sqrt{5}}{2\sqrt{2}\sqrt{\pi}}$	$-\frac{1}{2}$	5
Y	↑	$\frac{4}{3\sqrt{\pi}}$	$-\frac{1}{2}$	4	⚹	↑	$\frac{2\sqrt{3}}{3\sqrt{\pi}}$	$-\frac{1}{2}$	6
⋇	↑	$\frac{\sqrt{5}}{3\sqrt{2}\sqrt{\pi}}$	$-\frac{1}{2}$	5	✳	↑	$\frac{\sqrt{\frac{7}{3}}}{3\sqrt{\pi}}$	$-\frac{1}{2}$	7
⅄	↓	$\frac{12\sqrt{3}}{\pi}$	-2	$2\sqrt{3}$	⚹	↓	$\frac{\sqrt{3}(1+\sqrt{3})^{\frac{7}{2}}}{2\pi}$	-2	$2(1+\sqrt{3})$
×	↓	$\frac{12\sqrt{30}}{\pi}$	-2	$2\sqrt{6}$	✳	↓	$\frac{\sqrt{6(376+156\sqrt{6})(1+\sqrt{6})^{\frac{7}{2}}}}{5\sqrt{95}\pi}$	-2	$2(1+\sqrt{6})$
⅄	↓	$\frac{24\sqrt{2}}{\pi}$	-2	$2\sqrt{2}$	⚹	↓	$\frac{\sqrt{8}(1+\sqrt{2})^{\frac{7}{2}}}{\pi}$	-2	$2(1+\sqrt{2})$
⅄	·	$\frac{3\sqrt{3}}{2\sqrt{\pi}}$	$-\frac{3}{2}$	3	⚹	·	$\frac{3\sqrt{3}}{2\sqrt{\pi}}$	$-\frac{3}{2}$	6
⅄	·	$\frac{2\sqrt{2}}{\Gamma(\frac{1}{4})}$	$-\frac{3}{4}$	3	⅄	·	$\frac{3\sqrt{3}}{\sqrt{2}\Gamma(\frac{1}{4})}$	$-\frac{3}{4}$	3
⚹	·	$\frac{\sqrt{2}3^{\frac{3}{4}}}{\Gamma(\frac{1}{4})}$	$-\frac{3}{4}$	6	⅄	·	$\frac{8}{\pi}$	-2	4
⟋	·	$\frac{4\sqrt{3}}{3\Gamma(\frac{1}{3})}$	$-\frac{2}{3}$	4					

Table 1: Parameters for $q_{\mathfrak{S}}(n)$ for all non-isomorphic quarter plane classes with zero or vertical drift.

Proving the hypotheses

1. The growth of $1/4$-plane models is **bounded above** by $1/2$-plane models.
2. Explicit **lower bounds** are computed by reducing to 11 base cases using the following lemma.

Lemma: Let $d(j)$ be the number of Dyck prefixes of length j and let $w(i) \sim \kappa\beta^i i^\alpha$, where $\alpha \leq 0, \kappa, \beta \in \mathbb{R}^+$. Then:

$$w'(n) = \sum_{i \geq 0} \binom{n}{i} w(i) \bowtie (\beta+1)^n; \quad (1)$$

$$w''(n) = \sum_{i \geq 0} \binom{n}{i} w(i) d(n-i) \bowtie (\beta+2)^n. \quad (2)$$

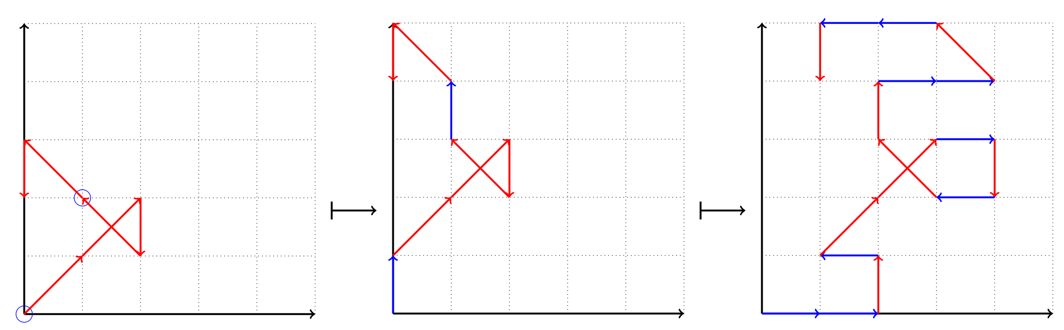


Figure 1: The number of ways of **inserting a single step not toward any boundary** is given by Equation (1) (first mapping), and Equation (2) gives the number of ways of **inserting a Dyck prefix on a pair of steps with drift away from any boundary** (second mapping).

Examples

We apply the methodology to a pair of examples. The first is a base case.

Proposition 1: $q_{\curlywedge}(n) \bowtie 2\sqrt{2}$.

1. **Upper bound** given by $1/2$ plane model

$$\lim \frac{1}{n} \log q_{\curlywedge}(n) \leq \log 2\sqrt{2}.$$

2. **Lower bound** found by counting walks returning to the origin. The count is a product of Catalan numbers $q_{\curlywedge}(0,0;4n) = C_{2n}C_n$, giving (since $C_n \bowtie 4$)

$$\log 2\sqrt{2} \leq \lim \frac{1}{n} \log q_{\curlywedge}(n).$$

The second uses our lemma to import a lower bound.

Proposition 2: $q_{\ast}(n) \bowtie 2(1+\sqrt{2})$.

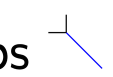
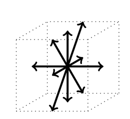
1. **Upper bound** given by $1/2$ plane model

$$\lim \frac{1}{n} \log q_{\ast}(n) \leq \log 2(1+\sqrt{2}).$$

2. **Lower bound** found by applying Equation (2) of the Lemma to the result of Proposition 1, giving

$$\log 2(1+\sqrt{2}) \leq \lim \frac{1}{n} \log q_{\ast}(n).$$

Perspective

- Unified approach - reduce the amount of case analysis.
- Generalise:
  1. More interesting drift: $\delta(\nearrow) = \swarrow$
  2. Bigger steps 
  3. Higer dimensional lattices 
- Understand the underlying singular behaviour of generating functions for $1/4$ plane models, à la [1].

References

[1] C. Banderier and P. Flajolet, *Basic analytic combinatorics of directed lattice paths*, Theoretical Computer Science, 2002.

[2] A. Bostan and M. Kauers, *Automatic classification of restricted lattice walks*, Discrete Mathematics and Theoretical Computer Science, 2009.

[3] M. Bousquet-Mélou, M. Mishna, *Walks with small steps in the quarter plane*, Contemporary Mathematics, Volume 520, 2010.