

On the irrationality of certain series

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Abstract

We prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{q^n + r} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{q^n + r}$$

are irrational and not Liouville whenever q is an integer ($q \neq 0, \pm 1$) and r is a non-zero rational ($r \neq -q^n$).

1. Introduction

In 1948, Erdős[5] proved the irrationality of the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1} = \sum_{n=1}^{\infty} \frac{d(n)}{2^n}$$

(here $d(n)$ is the divisor function) and some years later Erdős and Graham[7] speculated on the irrationality of $\sum_{n=1}^{\infty} (1/(2^n - 3))$. The author resolved this in [2] by proving $\sum_{n=1}^{\infty} (1/(q^n + r))$ irrational for q a positive integer (≥ 2) and r a non-zero rational. The proof is a moderately complicated application of Padé approximation methods. (See also [3].) It relies on detailed considerations of the Padé approximants in x to

$$\sum_{n=1}^{\infty} \frac{x^n}{q^n - 1} = \sum_{n=1}^{\infty} \frac{x}{q^n - x}.$$

This approach has been widely exploited and many, maybe most, irrationality proofs and estimates depend in one way or another on Padé approximants. See for example Chudnovsky and Chudnovsky[4], Mahler[9], Matala-Aho[10] and Walliser[11]. A further discussion of irrationality results for certain series is to be found in Erdős[6].

Our main results, Theorem 1 and Theorem 2, show that, for q an integer and r a non-zero rational, the following series are irrational:

$$\sum_{n=1}^{\infty} \frac{1}{q^n + r} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{q^n + r}$$

(provided that the series converge, so $q \neq 0, \pm 1, r \neq -q^n$). Theorem 1 extends the main theorem of [2] where we only handle the case when $q > 0$, and Theorem 2 is new. The proofs are both simpler and much more self-contained, needing only some

elementary complex analysis. The proofs in both cases rely on conjuring up the right contour integrals from which the results then naturally flow. While it is not logically necessary to see where these integrals come from, they are suggested by orthogonality considerations [8].

2. q -irrationalities

Our first theorem is

THEOREM 1. *Let q be an integer with $|q| > 1$ and let c be a non-zero rational number. Then*

$$\sum_{n=1}^{\infty} \frac{1}{q^n + c}$$

is irrational. (We assume that $c \neq -q^n$ for all n .)

The proof proceeds by considering the following contour integral:

$$F_n(q) = \frac{1}{2\pi i} \int_{|t|=1} \prod_{k=1}^{n-1} \left(\frac{1 - cq^k/t}{1 - q^k t} \right) \frac{(-1/t)}{(1 - q^n t)} \sum_{h=1}^{\infty} \frac{1}{1 - cq^h/t} dt. \quad (1)$$

Note that the integrand is meromorphic in t provided $|q| > 1$. We derive a number of lemmas. The first shows why we expect F_n to be an approximation to

$$\sum_{n=1}^{\infty} \frac{1}{1 - cq^n}.$$

LEMMA 1. *Suppose that $|q| > 1$ and $|c| > 1/|q|$. Then*

$$F_n(q) = \sum_{m=1}^n \frac{\prod_{k=1}^{n-1} (1 - cq^{k+m})}{\prod_{\substack{k=1 \\ k \neq m}}^{n-1} (1 - q^{k-m})} \left(\sum_{h=1}^{\infty} \frac{1}{1 - cq^h} - \sum_{h=1}^m \frac{1}{1 - cq^h} \right) + \frac{1}{(n-2)!} \frac{d^{n-2}}{dt^{n-2}} \left[\prod_{k=1}^{n-1} \left(\frac{t - cq^k}{1 - q^k t} \right) \frac{(-1)}{(1 - q^n t)} \sum_{h=1}^{\infty} \frac{1}{t - cq^h} \right]_{t=0}.$$

Proof. This is just the residue theorem. Note that the integrand in (1) has inside the circle $\{|t| = 1\}$ simple poles at $t = q^{-1}, q^{-2}, \dots, q^{-n}$. These give rise to the first terms above, on noting that

$$\sum_{h=1}^{\infty} \frac{1}{1 - cq^h q^m} = \sum_{h=1}^{\infty} \frac{1}{1 - cq^h} - \sum_{h=1}^m \frac{1}{1 - cq^h}. \quad (2)$$

The only other pole in $|t| \leq 1$ is a pole of order $n-1$ at $t = 0$ which accounts for the last term. ■

From (2) we see that, for rational c and q , the irrationality of

$$\sum_{h=1}^{\infty} \frac{1}{1 - (cq^m) q^h}$$

is equivalent to the irrationality of

$$\sum_{h=1}^{\infty} \frac{1}{1 - cq^h}.$$

So by replacing c by cq^m we may assume $|c| > 2$ (whenever $|q| > 1$). This we now do throughout the remainder of the paper.

The denominator polynomials $p_n(c, q)$ have the following explicit form.

LEMMA 2.

$$p_n(c, q) = \sum_{m=1}^n \frac{\prod_{k=1}^{n-1} (1 - cq^{k+m})}{\prod_{k=1}^{k+m} (1 - q^{k-m})} = \sum_{k=0}^{n-1} (-c)^k q^{k(k+3)/2} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \begin{bmatrix} n+k-1 \\ n-1 \end{bmatrix}_q$$

is a polynomial in c and q with integer coefficients and of degree $n-1$ in c .

Proof. That p_n is a polynomial of degree $n-1$ in c is obvious. The integrality in q follows from the second identity whose proofs relies on the Cauchy binomial theorem (we do not actually need this for the proof of irrationality, but with it we have better irrationality bounds). Here we have used the notation for the q -binomial coefficient

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{\prod_{i=1}^n (1 - q^i)}{\prod_{i=1}^m (1 - q^i)}$$

The Cauchy binomial theorem then asserts that

$$\sum_{m=0}^n y^m q^{m(m+1)/2} \begin{bmatrix} n \\ m \end{bmatrix}_q = \prod_{k=1}^n (1 + yq^k)$$

(See [1, p. 76].) ■

LEMMA 3 (Integrality of the form).

$$(n-2)! \prod_{k=1}^n (1 - cq^k) \prod_{k=\lceil n/2 \rceil}^n (1 - q^k) F_n(q) = (n-2)! \left(\prod_{k=1}^n (1 - cq^k) \prod_{k=\lceil n/2 \rceil}^n (1 - q^k) p_n(c, q) \right) \sum_{h=1}^{\infty} \frac{1}{1 - cq^h} + s_n(c, q)$$

where $s_n(c, q)$ is a polynomial in c and q , with integer coefficients, of degree at most $2n$ in c .

Proof. This follows easily from Lemma 1. One must consider evaluating the last term by repeated applications of the chain rule, and then one notes that

$$\left(\frac{d^{m-1}}{dt^{m-1}} \sum_{h=1}^{\infty} \frac{1}{t - cq^h} \right) \Big|_{t=0} = \frac{-(m-1)!}{c^m} \sum_{k=1}^{\infty} \frac{1}{q^{km}} = \frac{-(m-1)!}{c^m (q^m - 1)}$$

One should also note that if $0 \leq m \leq n$ then

$$(1 - q^m) \Big|_{\prod_{k=\lceil n/2 \rceil}^n (1 - q^k)} \quad \blacksquare$$

The error estimate we need is

LEMMA 4. For $|q| \geq 2$ and $|c| \geq 2$, one has $|F_n(q)| \leq 2^{n+1}/(q^{3/2n^2})$.

Proof. Let $\delta_1 = \{ |t| = 1 \}$ and $\delta_m = \{ |t| = |cq^m| + 1 \}$. Then

$$I_m := \frac{1}{2\pi i} \int_{\delta_1 \cup \delta_m} \prod_{k=1}^{n-1} \left(\frac{1 - cq^k/t}{1 - q^k t} \right) \frac{(-1/t)}{(1 - q^n t)} \left(\frac{1}{1 - cq^m/t} \right) dt = \left(\prod_{k=1}^{n-1} \frac{(1 - q^{k-m})}{(1 - cq^{k+m})} \right) \frac{-1}{(1 - cq^{m+n})},$$

since there is a single simple pole at $t = cq^m$. In particular, for $|q| \geq 2$, $|c| > 2$ and $m \geq n$,

$$|I_m| \leq \prod_{k=1}^n \frac{2}{|q|^{k+m}}.$$

So
$$|I_m| \leq \frac{2^n}{|q|^{mn+n^2/2}} \quad \text{and} \quad \sum_{m=n}^{\infty} |I_m| \leq \frac{2 \cdot 2^n}{|q|^{3n^2/2}}.$$

Since we can evaluate (1) by changing the contour to an arbitrarily large circle and evaluating at the intervening poles we have

$$|F_n(q)| = \left| \sum_{m=n}^{\infty} I_m \right|$$

and the proof is complete. (Note that the possible poles at cq^1, \dots, cq^{n-1} multiply out. Note also that the integral (1) vanishes at ∞ in the sense that the integrand in (1) tends to zero at least like $|cq^m|^{-2}$ on the circles δ_m of radius $|cq^m| + 1$, provided that $n > n_0$. This, coupled with convergence of the above series allows for the above evaluation.) **■**

Next we prove the non-vanishing of the form :

LEMMA 5. For $|q| \geq 2$ and $|c| \geq 1$ one has $F_n(q) \neq 0$ for all $n \geq n_0$.

Proof. As in Lemma 4,

$$I_m = - \prod_{k=1}^{n-1} \frac{(1 - q^{k-m})}{(1 - cq^{k+m})} \frac{1}{1 - cq^{m+n}}.$$

Now for $q > 1$ and $m \geq n$ the above quantity is negative if $c < -1$, and alternates if $c > 1$. In each case

$$F_n(q) = \sum_{m=n}^{\infty} I_m \neq 0,$$

because $|I_{m+1}| < |I_m|$ at least for n large. Similar considerations treat the case when $q < -1$. **■**

Proof of the Theorem. From Lemmas 3, 4, and 5 with

$$w_n(c, q) := (n-2)! \prod_{k=1}^n (1 - cq^k) \prod_{k=\lfloor n/2 \rfloor}^n (1 - q^k) p_n(c, q)$$

we have
$$0 < \left| w_n(c, q) \sum_{h=1}^{\infty} \frac{1}{1 - cq^h} + s_n c, q \right| \leq \frac{n! 4^{3n} |c|^n}{q^{n(n-2)5/8}}.$$

But if q is integral and $c = \alpha/\beta$ then $\beta^{2n} w_n(c, q)$ and $\beta^{2n} s_n(c, q)$ are integers, and the above error estimate when multiplied by β^{2n} still tends to zero, so that

$$\sum_{h=1}^{\infty} \frac{1}{1 - cq^h}$$

is irrational. **■**

Our second result is similar and we comment only on the details that differ.

THEOREM 2. *Let q be an integer with $|q| > 1$ and c a non-zero rational with $c \neq -q^n$ for all n . Then*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{q^n + c}$$

is irrational.

Proof. The contour integral to consider is now

$$\begin{aligned} F_n^*(q) &:= \frac{1}{2\pi i} \int_{|t|=1} \prod_{k=1}^{n-1} \frac{(1-cq^k/t)}{(1-q^k t)} \frac{(-1/t)}{(1-q^n t)} \sum_{h=1}^{\infty} \frac{(-1)^h}{1-cq^h/t} dt \\ &= \sum_{m=1}^n \frac{\prod_{k=1}^{n-1} (1-cq^{k+m})}{\prod_{k=1}^{k-1} (1-q^{k-m})} (-1)^m \left(\sum_{h=1}^{\infty} \frac{(-1)^h}{1-cq^h} - \sum_{h=1}^m \frac{(-1)^h}{1-cq^h} \right) \\ &\quad + \frac{1}{(n-2)!} \frac{d^{n-2}}{dt^{n-2}} \left[\prod_{k=1}^{n-1} \left(\frac{t-cq^k}{1-q^k t} \right) \left(\frac{-1}{1-q^n t} \right) \sum_{h=1}^{\infty} \frac{(-1)^h}{t-cq^h} \right]_{t=0}, \end{aligned}$$

and as before

$$|F_n^*(q)| \leq \frac{2 \cdot 2^n |c|^n}{q^{3/2n^2}}.$$

We now must multiply $F_n^*(q)$ by

$$(n-2)! \prod_{k=1}^n (1-q^k) \prod_{k=1}^n (1-cq^k) \prod_{k=1}^n (1+q^k)$$

to derive a form

$$G_n(q) := \alpha_n(c, q) \sum_{h=1}^{\infty} \frac{(-1)^h}{1-cq^h} + \beta_n(c, q),$$

where α_n and β_n have integer coefficients in c and q . The final term

$$\prod_{k=1}^n (1+q^k)$$

comes from the terms that arise from the pole at zero. Here we must evaluate

$$\sum_{h=1}^{\infty} \frac{(-1)^h}{cq^{hm}},$$

from which the terms $(1+q^k)$ come. In fact

$$\alpha_n(c, q) = (n-2)! \prod_{k=1}^n (1-q^k) \prod_{k=1}^n (1-cq^k) \prod_{k=1}^n (1+q^k) \gamma_n(c, q),$$

where

$$\gamma_n(c, q) = \sum_{m=1}^n (-1)^m \frac{\prod_{k=1}^{n-1} (1-cq^{k+m})}{\prod_{k=1}^{k-1} (1-q^{k-m})}.$$

The error estimate now becomes

$$0 < |G_n(q)| \leq \frac{n!D^n}{q^{n^2/18}}$$

for some $D := D_{q,c}$, while the non-vanishing of the form is essentially as in Lemma 5. \blacksquare

The estimates implicit in Theorems 1 and 2 are good enough to prove that the series in question are not Liouville, essentially because we can derive an asymptotic for the error in Lemma 4, and then all the estimated terms grow like $q^{\epsilon n^2}$. The estimates of the proof of Theorems 1 and 2, now by standard methods ([1], chapter 11), give that the numbers in question satisfy inequalities of the form

$$\left| \alpha - \frac{p}{q} \right| > \frac{1}{q^n} \quad \text{for all } p, q \in \mathbb{Z}$$

for some n and hence are not Liouville.

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