A Result Of Duffin and Schaeffer and Stuff

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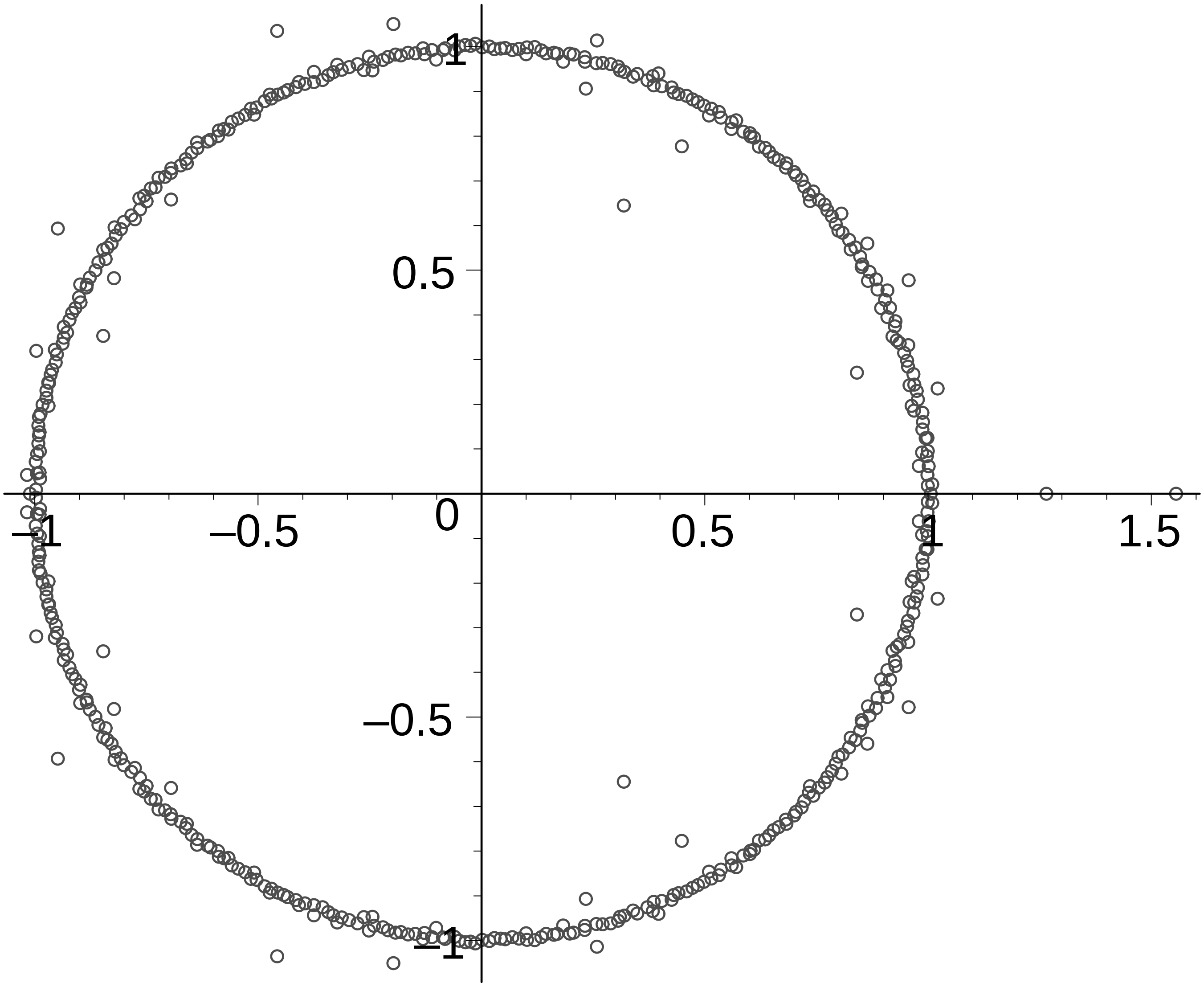
Abstract: Some results on zeros of partial sums.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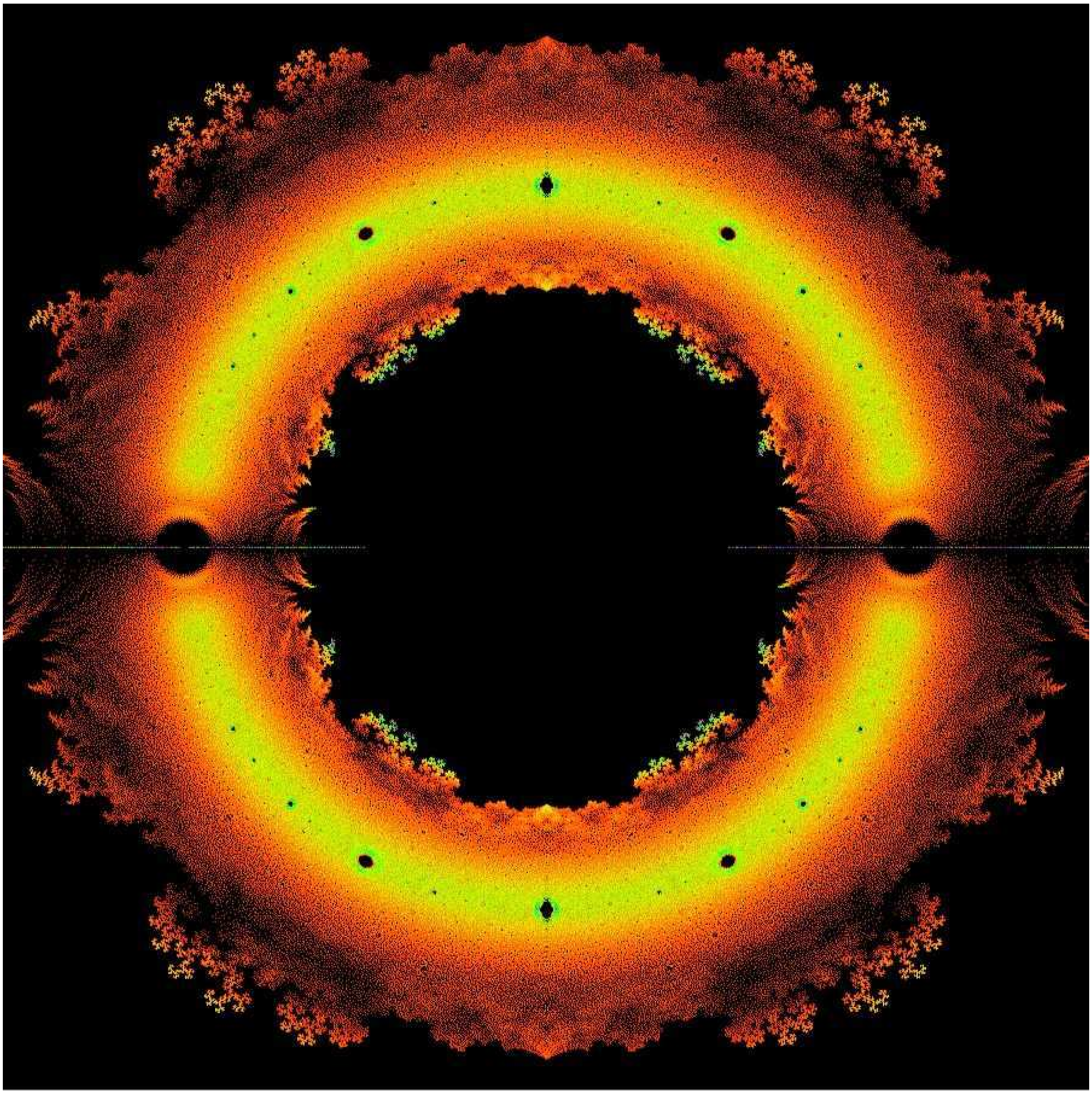

Introduction

Theorem 1 (Szegö (1922)) *A function $\sum_0^\infty a_n z^n$, where the a_n take only a finite number of different values, is rational if it can be continued analytically beyond the unit circle.*

Theorem 2 (Duffin & Schaeffer (1945)) *A function $\sum_0^\infty a_n z^n$, where the a_n take only a finite number of different values, is a rational function if it is bounded in a sector of the unit circle.*

Duffin and Schaeffer give the example

$$\sum_{1}^{\infty}(1-z)^n z^{n!}$$

of a function with integer coefficients which is bounded in the sector of the unit circle where $-\pi/4 \leq \arg z \leq \pi/4$ and which is not rational and has the unit circle as a natural boundary.

This follows because the terms satisfy the Fabry gap theorem.

Fabry's gap theorem

Let $f(z) = \sum_0^\infty c_n z^{\lambda_n}$. Let $\{\lambda_n\}$ be an increasing sequence of non-negative integers satisfying the Fabry gap condition: $\lambda_n/n \to \infty$.

Fabry's gap theorem asserts that, under this condition, every point on the boundary of the disk of convergence is a singular point.

Theorem of Pólya - proved by Carleson.

Theorem 3 (Pólya and Carlson (1921)) *A function $\sum_0^\infty a_n z^n$ which converges in the unit circle and where the a_n are integers is a rational function if it can be continued analytically beyond the unit circle.*

Theorem 4 (Jentzsch (1914)) *Each point of the circle of convergence of a power series $\sum_0^\infty a_n z^n$ having finite positive radius of convergence is a limit point of the set of zeros of the sequence of its partial sums.*

New Results and Proofs

We reprove a result:

Theorem 5 *Suppose* l_n *is a a sequence of polynomials with bounded integer coefficients (or with coefficients from a finite set). Then* l_n *cannot tend to zero uniformly on any arc on the unit circle.*

Proof. Consider

$$f(z) := \sum_{1}^{\infty} l_n(z) z^{\lambda_n}$$

where the l_n are polynomials with coefficients from a finite set and λ_n increases rapidly enough to satisfy the Fabry gap condition. Then we know from Duffin and Schaeffer (1945) that f cannot be bounded in any sector (or it would be rational and extend-able).

What does this tell us about the sequence l_n?

If the l_n tend to zero on a piece of arc then by judicial choices one can force a contradiction. Make a subsequence l'_n small enough so that

$$\sum_1^\infty l'_n(z)$$

converges uniformly for z on the arc. Now choose the λ_n growing rapidly enough to ensure the uniform convergence of

$$f(z) := \sum_1^\infty l'_n(z) z^{\lambda_n}$$

in the sector with the arc as boundary and also so that the sum has Fabry gaps.

Call Θ the class of analytic functions with coefficients just 0 or 1 in their Taylor series expansions.

Theorem 6 *If f is a rational function in Θ then*

$$f(z) := p_m(z) + z^{m+1}\frac{q_n(z)}{1 - z^{n+1}}$$

where p_m is a $0, 1$ polynomial of degree m and q_n is a $0, 1$ polynomial of degree n.

If f has no zeros in the unit disk then $(1 - z^{n+1})p_m(z) + (z^{m+1})q_n$ is cyclotomic.

Proof. Note that if f is a rational function than it is a quotient of two polynomials with rational coefficients. Since $f(1/2)$ is rational the binary expansion of $f(1/2)$ (which is just the coefficients of f) is ultimately periodic. The result follows immediately.

There is a similar result if the coefficients of f come from a bounded set of integers. See Pólya and G. Szegő, Volume 2, P158, Ex. 158.

Theorem 7 *Every infinite sequence of Littlewood polynomials has zeros dense in the unit circle.*

This holds for any sequence of polynomials with coefficients from any fixed finite set. This can also be deduced from Theorem 3.4.1 in [**?**] due to P. Erdős and P. Turán.

Proof. Suppose a sequence l_n of polynomials with $\{0, \pm 1\}$ coefficients (or coefficients from any fixed finite set) has no zeros in a a closed disk D centered at a point ζ on the unit circle. Suppose these polynomials do not vanish at 0. Let D_1 be a closed disk centered at ζ and contained in the interior of D.

Let m_n denote the (positive) minimum modulus of l_n on the boundary of D_1. We claim that m_n does not tend to zero and hence the l_n are bounded away from zero on D_1.

Suppose m_n does tend to zero. We can then assume, by passing to a subsequence if necessary, that there there exists a convergent subsequence of points ρ_n on the boundary of D_1 where $l_n(\rho_n)$ tends to zero. Let ρ denote the limit of the ρ_n then $l_n(\rho)$ tends to zero

Let $E := D \cap \{|z| \leq 1\}$ and $F := D \cap \{|z| \geq 1\}$.

We can assume, by passing to a subsequence if necessary that l_n converges to a power series f with $\{0, \pm 1\}$ coefficients and that the convergence is uniform on any compact subset of the interior of E.

We can also assume, by passing to a subsequence if necessary that l_n^* (here $*$ denotes the reciprocal polynomial) converges to a power series g with $\{0, \pm 1\}$ coefficients and that the convergence is uniform on any compact subset of the interior of F^*.

Note that $\rho \in D_1$ cannot be in the interior of either E or F since then the l_n would ultimately have zeros interior to D. It follows that there is a piece of arc in the interbf of E and F where l_n

trends to zero uniformly and this is impossible. (Take a piece of arc with endpoint ρ that extends toward the closest boundary point of $E \cap F$. To see the uniformity consider repeating the above argument on disks like D_1 with increasing radii and use the fact that the minimum modulus decreases on this larger set.)

We have shown that the sequence l_n is uniformly bounded below on compact subsets of D. Thus by Montel's theorem there is a subsequence of $\frac{1}{l_n}$

that converges to an analytic function on compact subsets of D and hence the same subsequence of l_n converges to to an analytic function on compact subsets of D. This would gives the contradiction that f has a convergent expansion with $\{0, \pm 1\}$ coefficients in a region outside the unit disc.

Theorem 8 (Vitali's Convergence Thm) *Let $f_n(z)$ be a sequence of functions, each regular in a region D, let $|f_n(z)| \leq M$ for every n and z in D, and let $f_n(z)$ tend to a limit as $n \to \infty$ at a set of points having a limit point inside D.*

Then $f_n(z)$ tends uniformly to a limit in any region bounded by a contour interior to D, the limit therefore being an analytic function of z.

Zeros at 1

Theorem 9 (An Inequality of Schur) *Suppose*

$$p(z) := \sum_{j=0}^{n} a_j z^j$$

has m *positive real roots. Then*

$$m^2 \le 2n \log \left(\frac{|a_0| + |a_1| + \cdots + |a_n|}{\sqrt{|a_0 a_n|}} \right).$$

Theorem 10 (Pólya and Szegő) *Suppose*

$$p(z) := \sum_{j=0}^{n} a_j z^j$$

has all its roots in the upper half plane $\Im(z) > 0$. *Suppose, for each k, the real and imaginary part of*

$$a_k := \alpha_k + i\beta_k.$$

Then

$$u(z) := \sum_{j=0}^{n} \alpha_j z^j$$

and

$$v(z) := \sum_{j=0}^{n} \beta_j z^j$$

have only real zeros.

Suppose

$$r(z) := c_0 + c_1 z + \ldots + c_{n-1} z^{n-1} + c_n z^n$$

has all real roots then

$$\sqrt{|c_0 c_n|}\, e^{n/4} \leq |c_0| + |c_1| + \cdots + |c_n|$$

or

$$\sqrt{|c_0 c_n|}\, (1.284)^n \leq |c_0| + |c_1| + \cdots + |c_n|.$$

Now suppose p is a Littlewood polynomial of even degree then

$$p(iz) := \pm 1 + c_1 z + \ldots + c_{n-1} z^{n-1} + \pm z^n$$

with each $|c_i| \leq 1$. (p Littlewood can be relaxed a lot.)

So p must have at least one root in the right half plane for $n > 9$ (whenever $(1.284)^n > n + 1$.)

A Better Estimate

Let

$$K_n :=$$

$$\left\{ \sum_{j=0}^{n} a_j x^j \,, \quad |a_0| = |a_n| = 1,\ |a_j| \le 1 \right\}.$$

Theorem 11 *Every $P \in K_n$ has at least $8\sqrt{n}\log n$ zeros in every open disk $D(z_0, \delta_n)$ centered at $z_0 \in \partial D$ with radius*

$$\delta_n := \frac{33\pi \log n}{\sqrt{n}}$$

whenever $\delta_n \le 1$.

The proof of the theorem follows from the combination of the two results below. The first one is a difficult result of Erdős and P. Turán,.

The second one is a consequence of Jensen's Formula.

Theorem 12 (Erdős and Turán) *If*

$$P(z) := \sum_{k=0}^{n} a_k z^k$$

and zeros

$$z_\nu = r_\nu \exp(i\varphi_\nu)\,, \qquad r_\nu > 0\,, \quad \varphi_\nu \in [0, 2\pi),$$

then for every $0 \le \alpha < \beta \le 2\pi$ *we have*

$$\left| \sum_{\nu \in I(\alpha,\beta)} 1 - \frac{\beta - \alpha}{2\pi} n \right| < 16\sqrt{n \log R}\,,$$

where

$$R := \frac{|a_0| + |a_1| + \cdots + |a_n|}{\sqrt{|a_0 a_n|}}$$

and

$$I(\alpha, \beta) := \{\nu \in \{1, 2, \ldots, n\} : \alpha \le \varphi_\nu \le \beta\}\,.$$

Lemma 1 *Let $\alpha \in (0,1)$. Every polynomial in K_n has at most*

$$(2/\alpha)\log(1/\alpha)$$

zeros in the open disk

$$D(0, 1-\alpha)$$

centered at the origin with radius $1-\alpha$ and outside the open disk

$$D(0, (1-\alpha)^{-1})$$

centered at the origin with radius $(1-\alpha)^{-1}$.

Proof of Theorem. Let $z_0 := e^{i\varphi}$. For $\gamma_n > 0$ let

$$S(z_0, \gamma_n) :=$$

$$\{re^{i\theta} : \quad 0 \leq r < \infty\,, \quad \varphi - \gamma_n < \theta < \varphi + \gamma_n\}\,.$$

Let

$$\gamma_n := \frac{32\pi \log n}{\sqrt{n}} \leq 1$$

Assume that $P \in K_n$ has m zeros in $\mathbb{C} \setminus S(z_0, \gamma_n)$.

It follows from Erdős and P. Turán that

$$m - \frac{2\pi - 2\gamma_n}{2\pi} n \leq 16\sqrt{n \log n},$$

That is,

$$m \leq n - \frac{\gamma_n n}{\pi} + 16\sqrt{n \log n}\,,$$

hence

$$m \leq n - 16\sqrt{n}\log n.$$

Thus P has at least $16\sqrt{n}\log n$ zeros in the sector $S(z_0, \gamma_n)$.

Now we use the above Lemma with $\alpha := n^{-1/2}$ to conclude the statement of the theorem.

Questions and stuff. Part 1

Question 1. Prove Duffin and Schaeffer directly for the class Θ.

That is prove

Theorem 13 *If f is in Θ and f is bounded in a sector of the unit circle then f is a rational function.*

Question 2. Is bounded just on a set of positive measure in the circle enough for Duffin and Schaeffer?

Question 3. Consider

$$f(z) := \sum_{1}^{\infty} l_n(z) z^{\lambda_n}$$

where the l_n are Littlewood polynomials (or 0, ± 1 polynomials) vanishing at 1, and λ_n increases rapidly enough to satisfy the Fabry gap condition. Then f cannot be bounded in any sector (or it would be rational and extendable).

What more does this tell us about the sequence l_n?

Question 4. What is the minimum number of zeros of modulus 1 of a real-valued Littlewood polynomial of degree n?

Littlewood [1966, problem 22] poses the following research problem, which appears to still be open: "If the n_m are integral and all different, what is the lower bound on the number of real zeros of $\sum_{m=1}^{N} \cos(n_m\theta)$? Possibly $N-1$, or not much less."

Question 5. What is the minimum number of zeros of modulus 1 of a reciprocal $0, 1$ (as above) polynomial.

Question 6. Suppose l_n is a sequence of Littlewood polynomials that is uniformly bounded in the half disk

$$\{|z| \le 1\} \cap \{\Re(z) \ge 0\}.$$

Show that a subsequence of l_n converges to a rational function.

This follows from DS. So prove directly.

Question 7. (Erdélyi) Does every Littlewood polynomiala of degree n have at least one zero in the annulus

$$\left\{1 - \frac{c}{n} < |z| < 1 + \frac{c}{n}\right\},$$

where $c > 0$ is an absolute constant.

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