Some Highly Computational Problems in Diophantine Number Theory

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Abstract: A number of classical and not so classical problems in number theory concern finding polynomials with integer coefficients that are of small norm. These include some old chestnuts like the Tarry-Escott problem and Littlewood's (other) Conjecture.

Let

$$\mathcal{Z}_n := \left\{ \sum_{i=0}^{n} a_i z^i : a_i \in \mathbb{Z} \right\}$$

denote the set of algebraic polynomials of degree at most n with integer coefficients and let $\mathcal{Z}$ denote the union over n of all such polynomials.

Let

$$\mathcal{F}_n := \left\{ \sum_{i=0}^{n} a_i z^i : a_i \in \{-1, 0, 1\} \right\}$$

denote the set of polynomials of degree at most n with coefficients from $\{-1, 0, 1\}$. These are the polynomials of degree n of height 1.

Let

$$\mathcal{L}_n := \left\{ \sum_{i=0}^{n} a_i z^i : a_i \in \{-1, 1\} \right\}$$

denote the set of polynomials of degree at most n with coefficients from $\{-1, 1\}$. In general we will call polynomials with coefficients $\{-1, 1\}$ **Littlewood polynomials**.

P1. The Integer–Chebyshev Problem. *Find the polynomial in $\mathcal{Z}_n$ that has smallest possible supremum norm on the unit interval. Analyse the asmptotic behaviour as n tends to infinity .*

Let

$$C_N[\alpha,\beta] :=$$

$$\left(\min_{a_i \in z, a_N \neq 0} \|a_0 + a_1 x + \ldots + a_N x^N\|_{[\alpha,\beta]}\right)^{\frac{1}{N}}.$$

One can show that

$$C[\alpha,\beta] := \lim_{N \to \infty} C_n[\alpha,\beta]$$

exists. This is the integer Chebyshev constant for the interval or the integer transfinite diameter. One can also show that

$$\frac{1}{2.3768-\in} \leq C[0,1] \leq \frac{1}{2.360}.$$

Lemma *Suppose*

$$q_m(x) = a_m x^m + \ldots + a_0, \qquad a_m \in Z$$

has all its roots in $(0,1)$. *(That is:* $q_m \in TR(0,1)$*). Provided* $(q_m, p_n) = 1$

$$\|p_n\|_{[0,1]}^{1/n} \geq \frac{1}{a_m^{1/m}}.$$

It is conjectured (Chudnovskys, Montgomery) that the lemma gives the right bound. (This is likely false.)

P2. The Tarry–Escott Problem. *Find the polynomial of with integer coefficients that is divisible by* $(x-1)^n$ *and has smallest possible* l_1 *norm. (That is, minimize the sum of the absolute values of the coefficients.)*

Almost equivalently (though not quite obviously) this polynomial must have coefficients $\{0, -1, +1\}$ and so

$$\|p\|_{L^2\{|z|=1\}} = \sqrt{2n}.$$

The Basis for the Conjecture

$$x^{\alpha_1}+\ldots+x^{\alpha_N}-x^{\beta_1}-\ldots-x^{\beta_N} = 0((x-1)^N).$$

For $N = 3, \ldots, 10$ with

$$[\alpha_1, \ldots, \alpha_N] \quad \text{and} \quad [\beta_1, \ldots, \beta_N]$$

- $[1, 2, 6] = [0, 4, 5]$
- $[0, 4, 7, 11] = [1, 2, 9, 10]$
- $[1, 2, 10, 14, 18] = [0, 4, 8, 16, 17]$
- $[0, 4, 9, 17, 22, 26] = [1, 2, 12, 14, 24, 25]$

- $[0, 18, 27, 58, 64, 89, 101]$
$= [1, 13, 38, 44, 75, 84, 102]$

- $[0, 4, 9, 23, 27, 41, 46, 50]$
$= [1, 2, 11, 20, 30, 39, 48, 49]$

- $[0, 24, 30, 83, 86, 133, 157, 181, 197]$
$= [1, 17, 41, 65, 112, 115, 168, 174, 198]$

- $[0, 3083, 3301, 11893, 23314, 24186,$
$35607, 44199, 44417, 47500] =$

$[12, 2865, 3519, 11869, 23738, 23762,$
$35631, 43981, 44635, 47488]$

New Size 10 and 12 solutions

- $[0, 12, 125, 213, 214, 412, 413, 501, 614, 626]$
$= [5, 6, 133, 182, 242, 384, 444, 493, 620, 621]$

- $[-515, -452, -366, -189, -103, 103, 189, 366, 452, 515]$
$= [-508, -471, -331, -245, -18, 18, 245, 331, 471, 508]$

- $[0, 11, 24, 65, 90, 129, 173, 212, 237, 278, 291, 302]$
$= [3, 5, 30, 57, 104, 116, 186, 198, 245, 272, 297, 299]$

Diophantine Form

Find distinct integers

$[\alpha_1, \ldots, \alpha_N]$ and $[\beta_1, \ldots, \beta_N]$

so that

$$\alpha_1 + \ldots + \alpha_N = \beta_1 + \ldots + \beta_N$$

$$\alpha_1^2 + \ldots + \alpha_N^2 = \beta_1^2 + \ldots + \beta_N^2$$

$$\vdots$$

$$\alpha_1^{N-1} + \ldots + \alpha_N^{N-1} = \beta_1^{N-1} + \ldots + \beta_N^{N-1}$$

The problem is completely open for $N \geq 13$.

P3. Erdős and Szekerez Problem. *For each n minimize*

$$\|(1-z^{\alpha_1})(1-z^{\alpha_2})\ldots(1-z^{\alpha_n})\|_D$$

where the α_i are positive integers. In particular show that these mins grow faster than β^n for any positive constant β.

Call this minimum S_N. From the PTE problem

$$S_N \geq 2N$$

Examples

N	$\|f\|_1$	$\{\alpha_1, \dots, \alpha_N\}$
1	2	$\{1\}$
2	4	$\{1, 2\}$
3	6	$\{1, 2, 3\}$
4	8	$\{1, 2, 3, 4\}$
5	10	$\{1, 2, 3, 5, 7\}$
6	12	$\{1, 1, 2, 3, 4, 5\}$
7	16	$\{1, 2, 3, 4, 5, 7, 11\}$
8	16	$\{1, 2, 3, 5, 7, 8, 11, 13\}$
9	20	$\{1, 2, 3, 4, 5, 7, 9, 11, 13\}$
10	24	$\{1, 2, 3, 4, 5, 7, 9, 11, 13, 17\}$
11	28	$\{1, 2, 3, 5, 7, 8, 9, 11, 13, 17, 19\}$
12	36	$\{1, \dots, 9, 11, 13, 17\}$
13	48	$\{1, \dots, 9, 11, 13, 17, 19\}$

Conjecture Except for $N = 1, 2, 3, 4, 5, 6$ and 8

$$S_N \geq 2N + 2.$$

- Maltby solves this for N=7, 9 and 10.

P4. Littlewood's Problem in L_∞. *Find the polynomial in $\mathcal{L}_n$ that has smallest possible supremum norm on the unit disk. Show that there exist positive constants c_1 and c_2 so that for any n is it is possible to find $p_n \in \mathcal{L}_n$ with*

$$c_1\sqrt{n} \le |p_n(z)| \le c_2\sqrt{n}$$

for all complex z with $|z| = 1$.

Littlewood, in part, based his conjecture on computations of all such polynomials up to degree twenty.

Odlyzko has now done 200 MIPS years of computing on this problem

P5. Erdős's Problem in L_∞. *Show that there exists a positive constant c_3 so that for all n and all $p_n \in \mathcal{L}_n$ we have $\|p_n\|_D \geq (1 + c_3)\sqrt{n}$.*

P6. The Merit Factor Problem. *Find the polynomial in $\mathcal{L}_n$ that has smallest possible L_4 norm on the unit disk. Show that there exists a positive constant c_4 so that for all n and all $p_n \in \mathcal{L}_n$ we have $L_4(p_n) \geq (1 + c_4)\sqrt{n}$.*

P7. The Barker Polynomial Problem. *For $n > 12$ and $p_n \in \mathcal{L}_n$ $L_4(p_n) > ((n+1)^2 + 2n))^{1/4}$.*

A Barker polynomial

$$p(z) := \sum_{k=0}^{n} a_k z^k$$

with each $a_k \in \{-1, +1\}$ so that

$$p(z)\overline{p(z)} := \sum_{k=-n}^{n} c_k z^k$$

satisfies $c_0 = n + 1$ and

$$|c_j| \leq 1, \qquad j = 1, 2, 3 \ldots .$$

If $p(z)$ is a Barker polynomial of degree n then

$$||p||_4 \leq ((n+1)^2 + 2n))^{1/4}$$

P8. Lehmer's Problem (1933). *Show that a (non-cyclotomic) polynomial p with integer coefficients has Mahler measure at least* 1.1762.... *(This latter constant is the Mahler measure of* $1 + z - z^3 - z^4 - z^5 - z^6 - z^7 + z^9 + z^{10}$*.)*

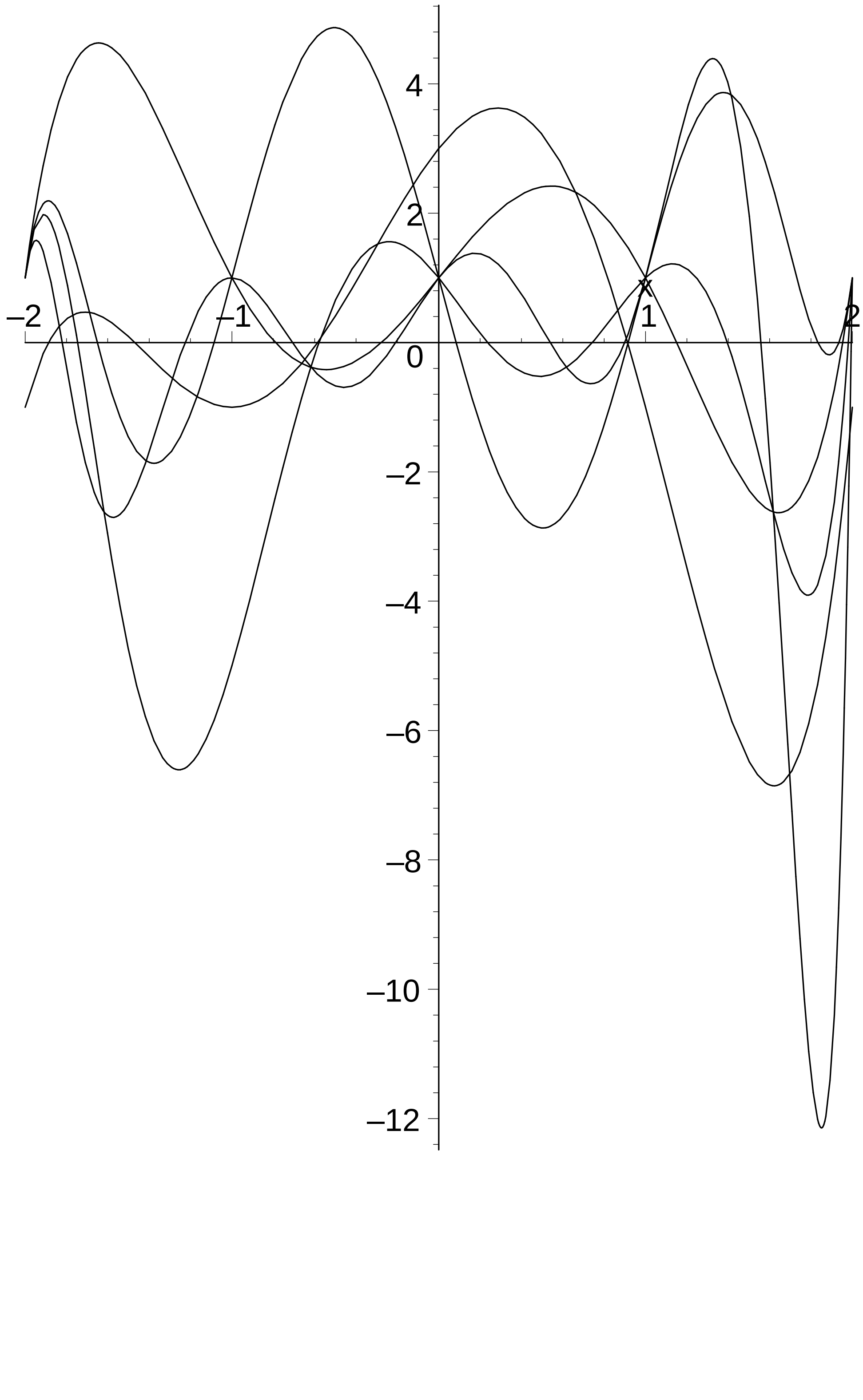
A conjecture of similar flavour is

P12. Conjecture of Schinzel and Zassenhaus (1965). *There is a constant c so that any non-cyclotomic polynomial p_n of degree n with integer coefficients has at least one root of modulus at least c/n.*

This conjecture is made in Schinzel and Zassenhaus [1965]. It is easy to see that P8 implies P12. The best partial is due to Smyth. If p is a non-reciprocal polynomial of degree n then at least one root ρ satisfies

$$\rho \geq 1 + \frac{\log \phi}{n}$$

where $\phi = 1.3247\ldots$ is the smallest Pisot number, namely the real root of $z^3 - z - 1$.


4
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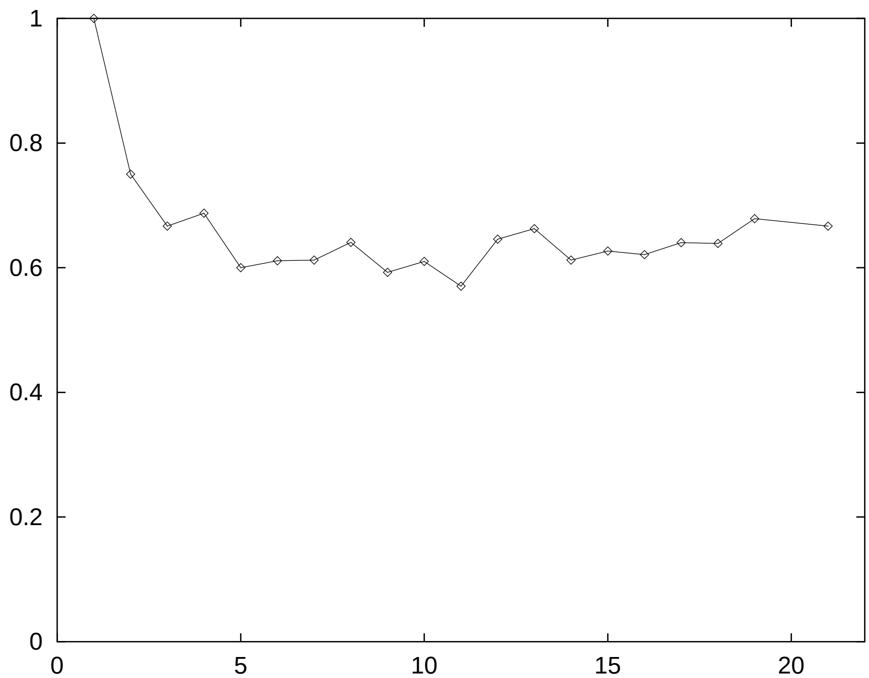
P9. Mahler's Problem. *For each n find the polynomials in* $\mathcal{L}_n$ *that have largest possible Mahler measure. Analyse the asymptotic behaviour as* n *tends to infinity.*

P10. Multiplicity of Zeros of Height One Polynomials. *What is the maximum multiplicity of the vanishing at 1 of a polynomial in* $\mathcal{F}_n$*?*

P11. Muliplicity of Zeros in $\mathcal{L}_n$**.** *What is the maximum multiplicity of the vanishing at 1 of a polynomial in* $\mathcal{L}_n$*?*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