Millenium Problem: P $\neq$ NP

David Laferrière

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- in general, receives *input* and provides results as a final state called *output*
- must terminate after a finite number of instructions

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procedure SumFirstN (n: positive integer) begin
  x:=0
  for i:=1 to n do
    x:=x+i
end
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So SumFirstN(5) would output the integer 15.

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- Given two or more algorithms that solve the same problem, can we decide which is faster?
- Of course, the answer to both question is "Yes"

Time-Complexity Function

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But how do we relate the time-complexity functions of two different algorithms?

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- Instead we use *big-oh* notation

Big-oh notation

Definition

Let $f, g : \mathbb{N} \to \mathbb{R}$. We say that g *dominates* f if there exist constants $m \in \mathbb{R}^+$ and $k \in \mathbb{N}$ such that $|f(n)| \leq m|g(n)|$ for all $n \in \mathbb{N}, n \geq k$.

When f is dominated by g, we denote this by $f(n) \in O(g(n))$.

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- Cook gave the first proof that a problem was NP-complete in 1971
- Karp in 1972 presented a collection of other NP-complete problems

What is a Problem?

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A *decision problem* Π is a set D_Π of *instances* subject to a question and a subset $Y_\Pi \subseteq D_\Pi$ of *yes-instances*.

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- P is the class of so-called "tractable" problems

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- NP is the complexity class of decision problems that can be solved by a non-deterministic Turing machine (NDTM) in polynomial time
- NP is equivalently the class of decision problems whose solutions can be verified in polynomial time

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  - Returns "yes," returns "no," or does not halt

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Denoted $\Pi_1 \propto \Pi_2$.

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A decision problem is *NP-Complete* if it is in *NP* and it is NP-Hard.

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- A *clause* over U is a set of literals over U
- A clause is *satisfied* by a truth assignment iff at least one of its members is true under that assignment
- A collection C of clauses over U is *satisfiable* iff there exists some truth assignment for U that simultaneously satisfies each of the clauses in C

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Let $U = \{u_1, u_2, u_3\}$, and let $t(u_1) = T$, $t(u_2) = T$, and let $t(u_3) = F$. Then $\{u_1, u_3\}$ is satisfied by t.
$C = \{\{u_1, \overline{u}_2\}, \{\overline{u}_1, u_2\}\}$ is satisfiable.
$C' = \{\{u_1, u_2\}, \{u_1, \overline{u}_2\}, \{\overline{u}_1\}\}$ is *not* satisfiable.

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QUESTION: Is there a satisfying truth assignment for C?

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Proof.

It is easy to see that satisfiability is in NP. The other direction is much more difficult (six pages in Garey and Johnson). $\square$

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Question: Is there a tour of all cities in C having total length no more than B?

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Question: Does G contain a Hamiltonian circuit?

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First we require a function f that maps each instance of HC to a corresponding instance of TS that satisfies the two properties required of a polynomial reduction.

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First we require a function f that maps each instance of HC to a corresponding instance of TS that satisfies the two properties required of a polynomial reduction.

Define f as follows: suppose $G = (V, E)$, with $|V| = m$, is an instance of HC. The corresponding instance of TS has a set C of cities that is identical to V. For any two cities $v_i, v_j \in C$, we define the inter-city distance $d(v_i, v_j)$ as follows:

$$d(v_i, v_j) = \begin{cases} 1 & \text{if } \{v_i, v_j\} \in E \\ 2 & \text{otherwise.} \end{cases}$$

Example

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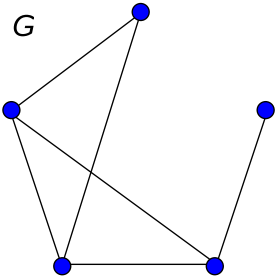

- Begin with a graph on five vertices.

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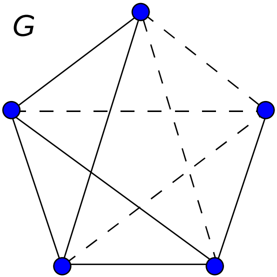

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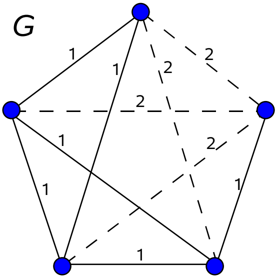
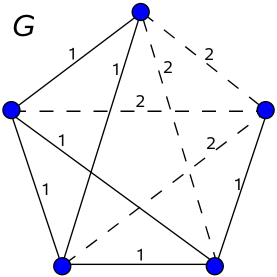


- Begin with a graph on five vertices.
- Make HC instance into TSP.
- Weight each edge.

Example



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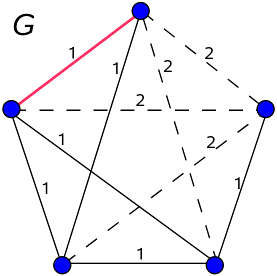

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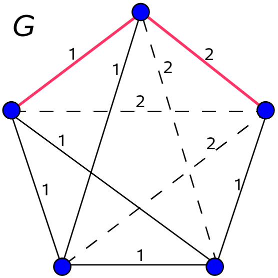

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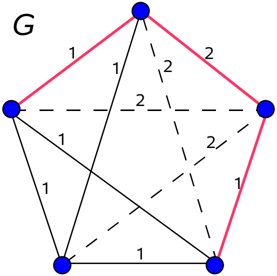

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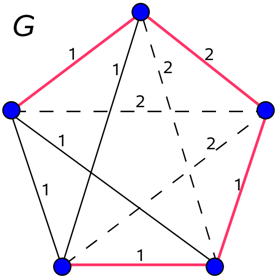

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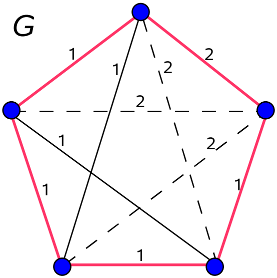

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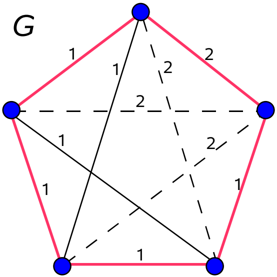

- Try to find a cycle of weight ≤ 5.
- None exist.

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If we have a tour of weight m, we must have a HC.

$\square$

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Does $P = NP$?

Possible Answers

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- $P = NP$ via existence proof
- $P = NP$ via constructive proof

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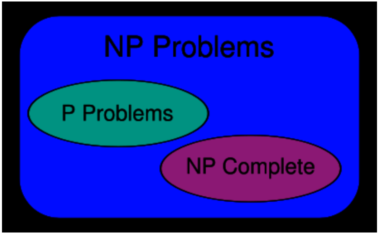

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- Practical algorithm for solving NP-complete problems would fundamentally change mathematics
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  - We could potentially solve *all* the Millenium Problems

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- The more we learn, the further we seem from a solution
- Razborov and Rudich (1993) showed that, given a certain credible assumption, "natural" proofs cannot solve $P = NP$
- Potential solution is a polynomial-time algorithm which solves an NP-complete problem

Thank You.