# Fast Computation of the Riemann Zeta Function to Arbitrary Accuracy

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# The Players

$$s = \sigma + it$$

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}\ , \quad \sigma > 1$$

$$\chi(s) = \frac{\zeta(s)}{\zeta(1-s)}$$

$$= \pi^{s-1/2}\ \frac{\Gamma((1-s)/2)}{\Gamma(s/2)}$$

Note $\chi(s)$ has poles at $s = 1, 3, \ldots 2n{+}1, \ldots$

$$Z(t) = \frac{\zeta(1/2+it)}{\sqrt{\chi(1/2+it)}}$$

If $t \in \mathbb{R}$ then $|Z(t)| = |\zeta(1/2+it)|$ and $Z(t) \in \mathbb{R}$ .



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# Motivation

The Lagarias-Odlyzko "analytic algorithm" for $\pi(x)$ [LO87, Gal99] requires many precise values of $\zeta(\sigma + it)$, with $\sigma$ fixed, $\sigma > 1$. E.g. to compute $\pi(10^{24})$ it would suffice to compute $\zeta(1.2 + ikh)$ to roughly $45$ digit precision, with

$$h = 0.0075$$
$$0 \leq k \leq 1.6 \times 10^{15}$$
$$(\text{i.e. } 0 \leq t \leq 1.2 \times 10^{13})$$



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# Euler-Maclaurin Summation

$$\zeta(s) = \sum_{n=1}^{N} n^{-s} + \frac{N^{1-s}}{s-1}$$
$$+ \sum_{m=1}^{M-1} \frac{B_m}{m} \binom{s+m-2}{m-1} N^{1-m-s}$$
$$+ O_M(s^M N^{1-M-s})$$

## Advantages

- Error analysis is straightforward [CO92].
- Allows arbitrary accuracy.
- $O(t^{1+\epsilon} + d^{1+\epsilon})$ operations to find $d$ digits.

## Disadvantages

- $N \gtrapprox t$.
- Needs Bernoulli numbers.



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# Riemann-Siegel Formula

$$Z(t) = 2\sum_{n=1}^{N} n^{-1/2-it}/\sqrt{\chi(1/2+it)}$$
$$+\sum_{m=0}^{M-1} \cdots + O(t^{-1/4-M/2}) \quad \text{as } t \to \infty,$$

where $N = \lfloor \sqrt{t/(2\pi)} \rfloor$.

**Advantages**

- $N \approx \sqrt{t/(2\pi)}$ for large $t$.

**Disadvantages**

- Error analysis is difficult (currently only available for $\sigma = 1/2$) [Gab79].
- Accuracy is limited.



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# Quadrature Near Saddle Point

Use numerical quadrature instead of the asymptotic expansion developed in the Riemann-Siegel formula:

**Advantages**

- $N \approx \sqrt{t/(2\pi)}$ for large $t$.
- Error analysis is straightforward.
- Allows arbitrary accuracy.

**Disadvantages**

- $O(t^{1/2+\epsilon} + d^{3/2+\epsilon})$ operations to find $d$ digits. [The Euler-Maclaurin formula wins when *very* high accuracy is needed.]



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# An Integral for $\zeta(s)$

Let

$$I_N(s) = \int_{N \nearrow N+1} \frac{\exp(i\pi z^2)\ z^{-s}}{e^{i\pi z} - e^{-i\pi z}}\, dz,$$

then

$$\zeta(s) = I_0(s) + \chi(s)\ \overline{I_0(1-\overline{s})}.$$

We then use

$$I_0(s) = \sum_{n=1}^{N} n^{-s} + I_N(s)$$

and choose $N$ so the path is near the saddle point of $\exp(i\pi z^2)z^{-s}$ at $z = \sqrt{s/(2\pi i)}$.



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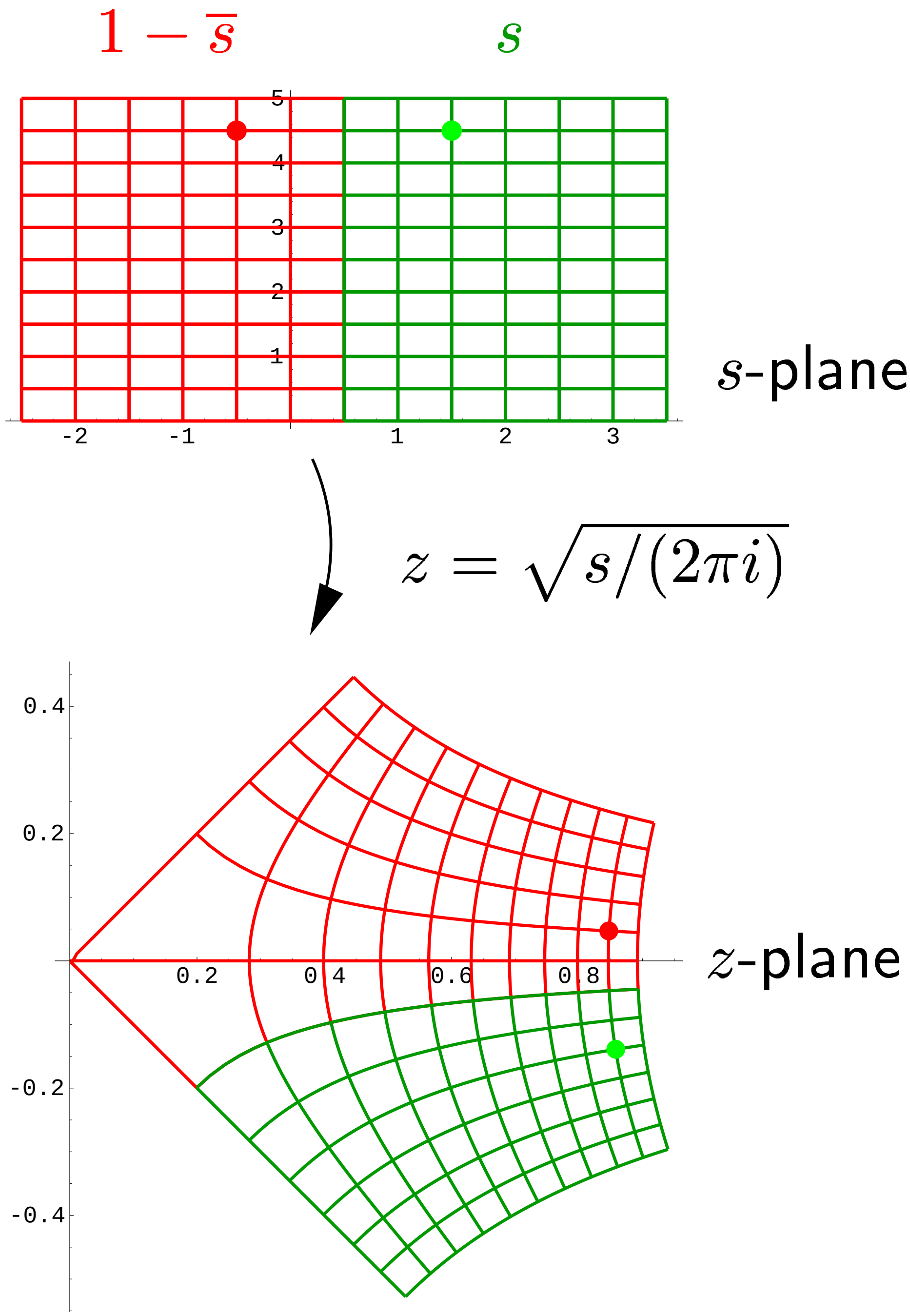
# Location of saddle point(s)

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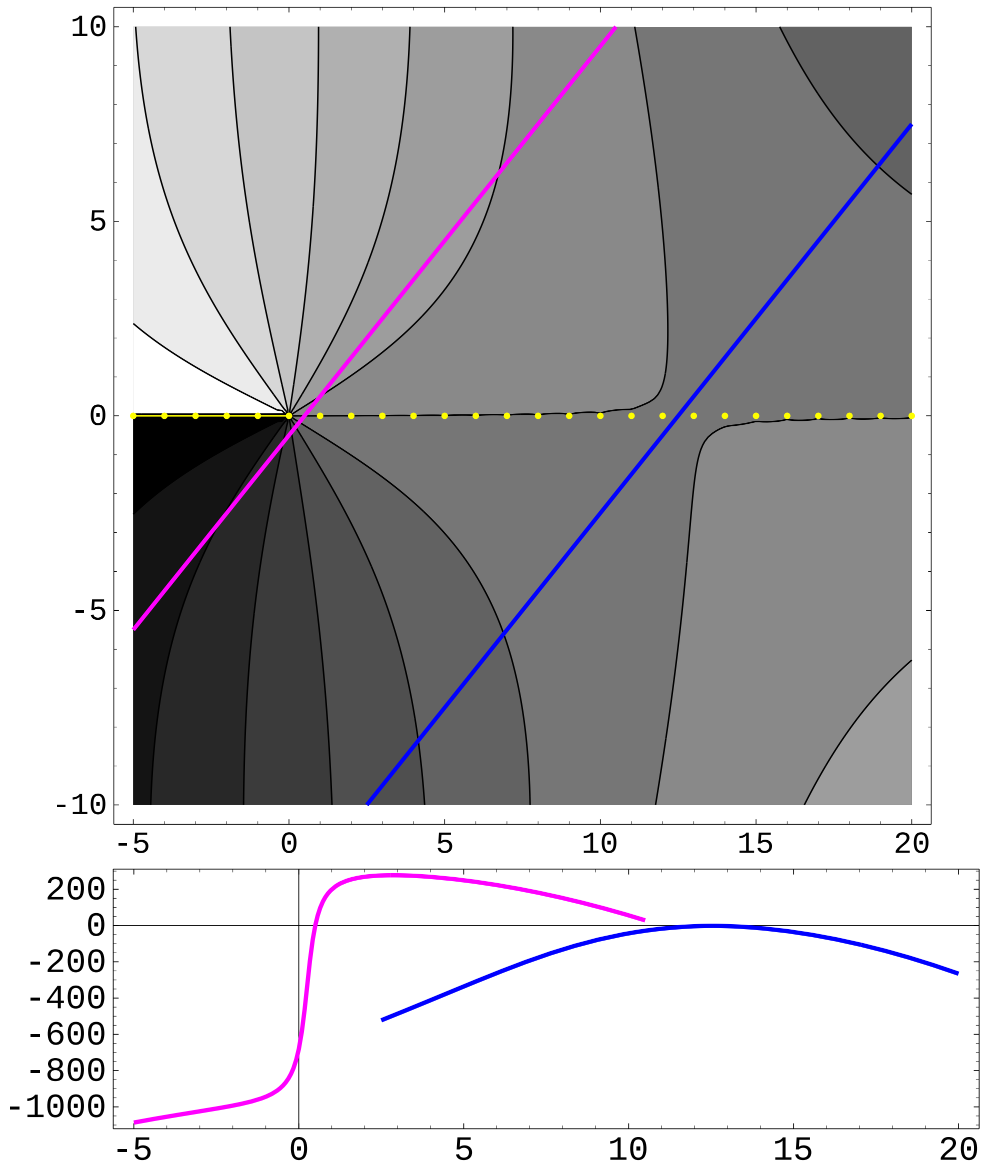
$$\log \left| \frac{\exp(i\pi z^2)\ z^{-s}}{e^{i\pi z} - e^{-i\pi z}} \right|$$




$$s = 1.5 + 1000i,\ N = 12$$

$0 \nearrow 1$ and $12 \nearrow 13$

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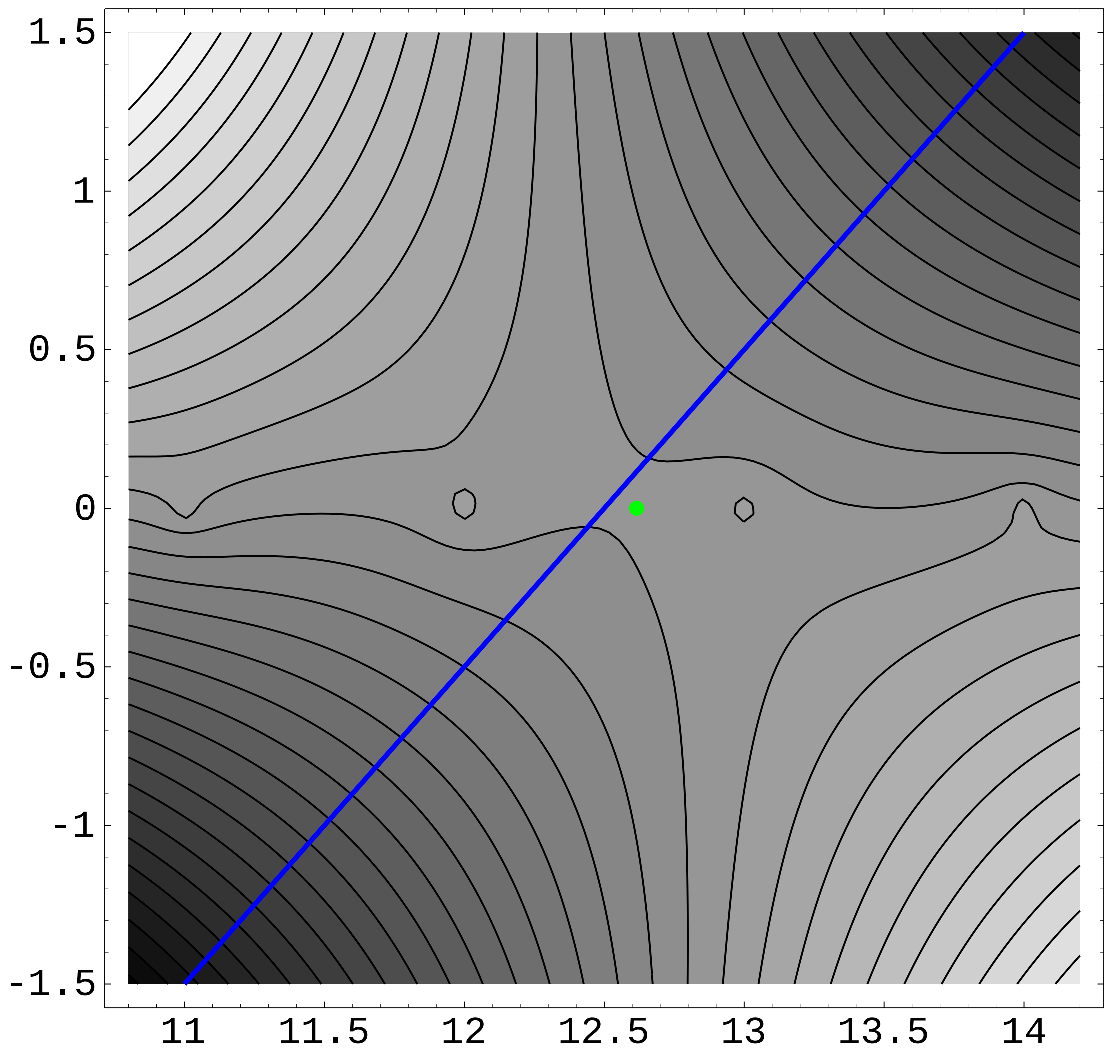
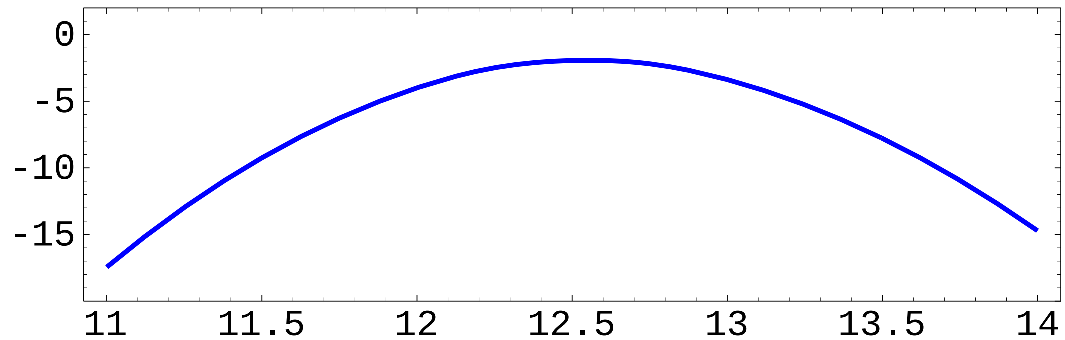
# Closeup View



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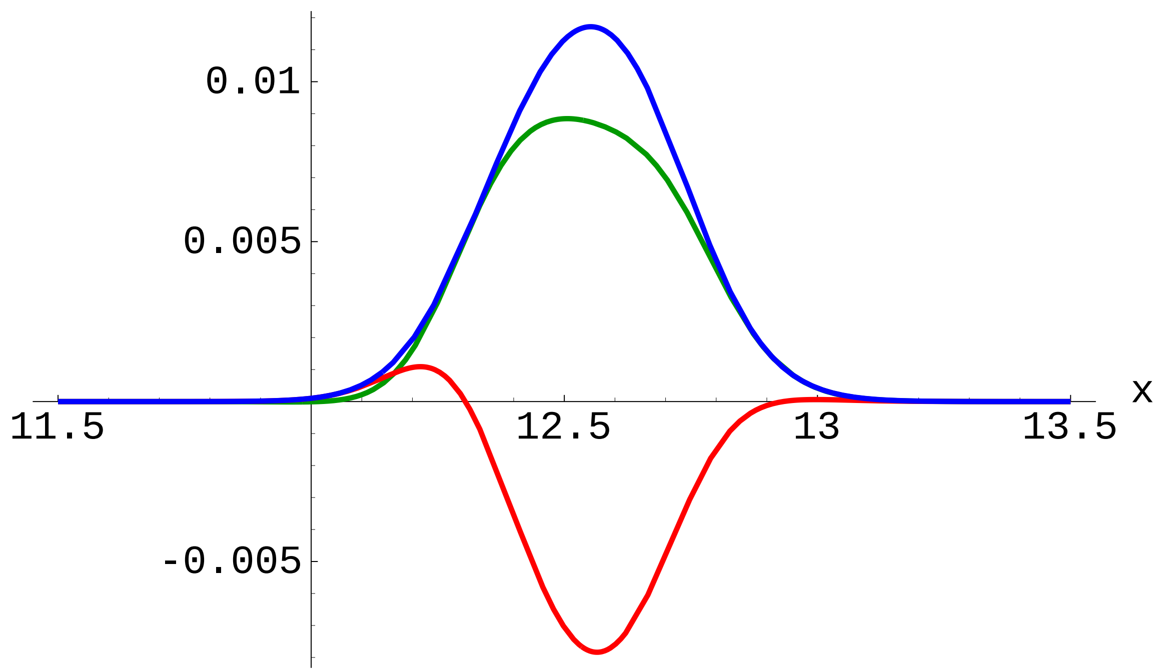
# The Integrand

Absolute Value, Real and Imaginary Parts of

$$f(s,z) = \frac{\exp(i\pi z^2)\ z^{-s}}{e^{i\pi z} - e^{-i\pi z}}$$

$$z = x + i(x - 12.5)$$

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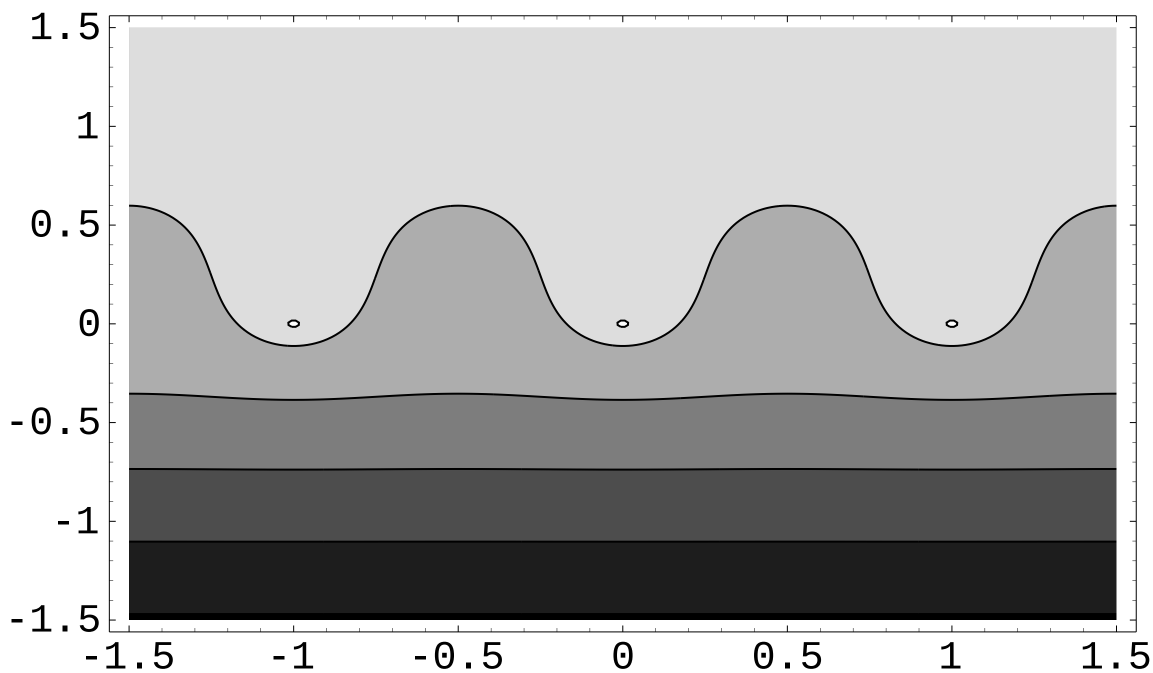
# Quadrature Analysis — a Tool

Let $H(w) = \dfrac{1}{1 - e^{2i\pi w}}$.

Note $H(u - iv) = O(e^{-2\pi v})$ as $v \to \infty$.



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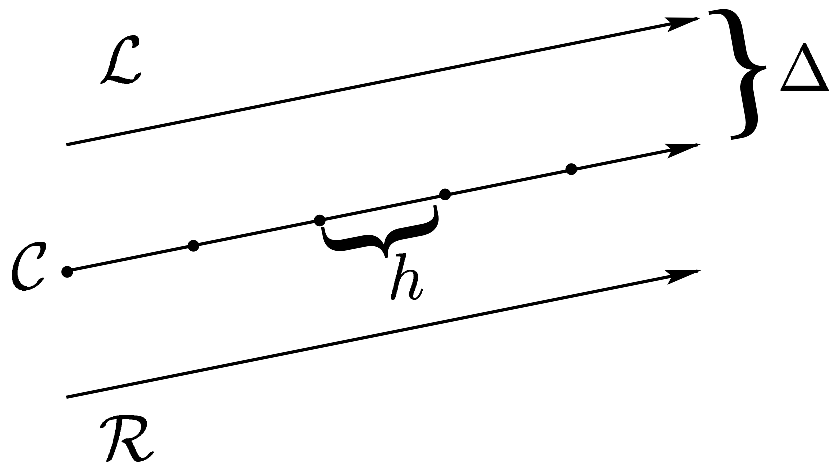
# A Quadrature Formula




$$\int_{\mathcal{C}} f(z)\,dz = h \sum_{m\in\mathbb{Z}} f(z_0 + m\,h) + \int_{\mathcal{L}} f(z) H((z_0 - z)/h)\,dz + \int_{\mathcal{R}} f(z) H((z - z_0)/h)\,dz,$$

Note $\int_{\mathcal{L}} = O_\Delta(e^{-2\pi\Delta/|h|})$, similarly for $\int_{\mathcal{R}}$.



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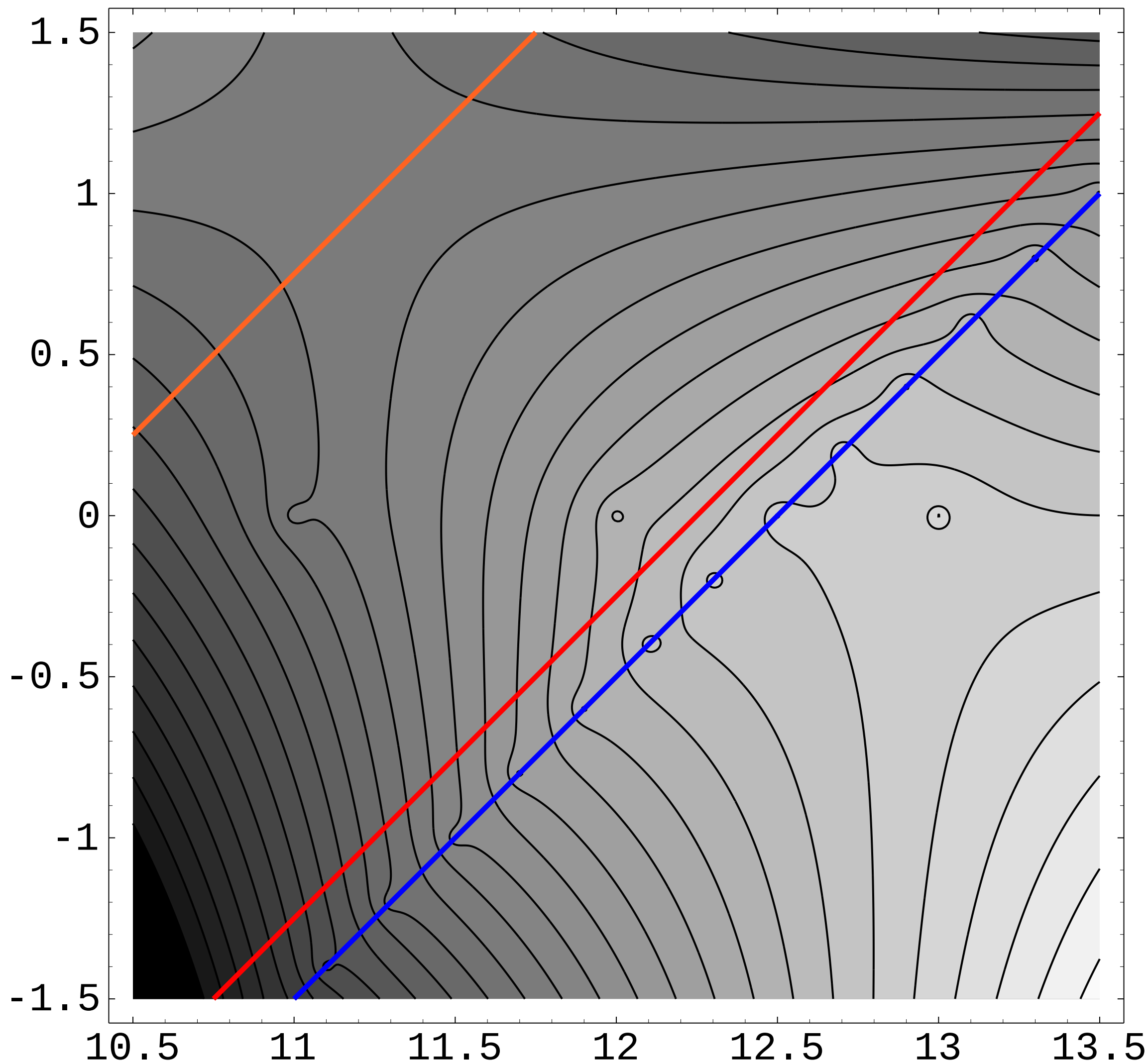
# “Left Error” for $I_0$ Integrand




$$h = 0.2(1+i)$$

$10 \nearrow 10.5$, $12 \nearrow 12.5$ and $12 \nearrow 13$



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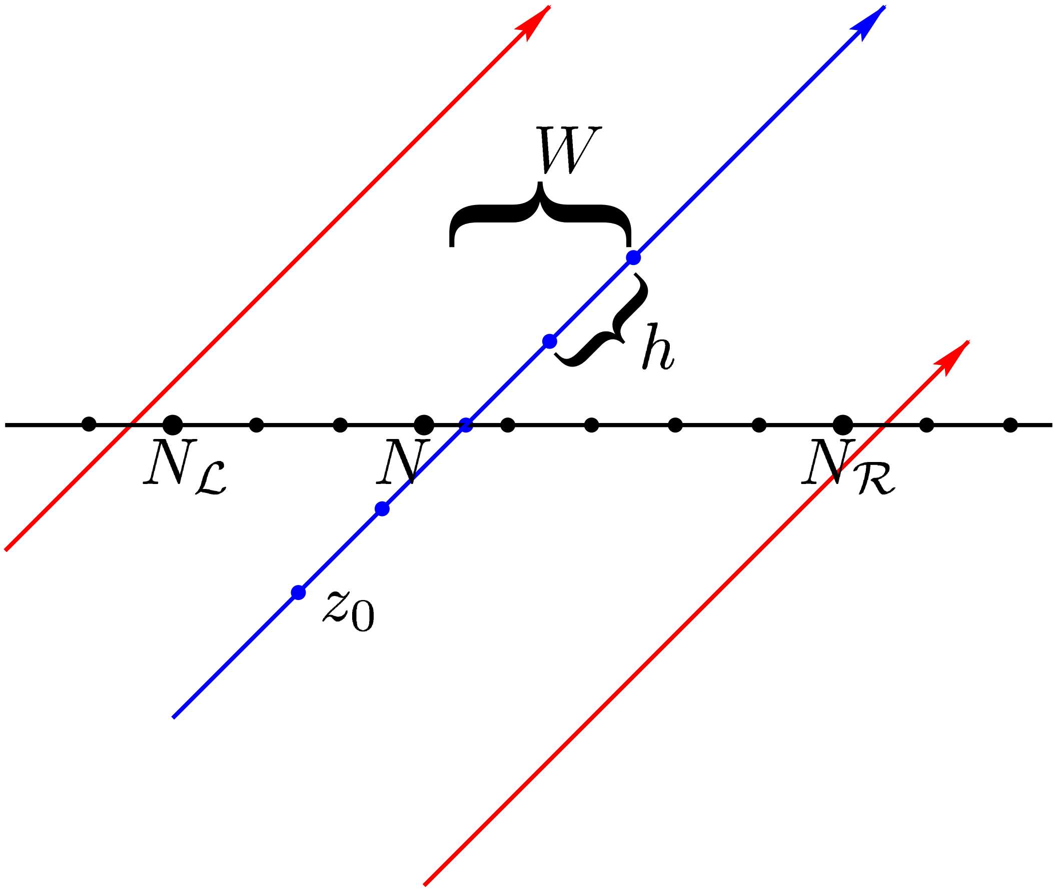
# 5 Parameters for Quadrature




$N_{\mathcal{L}}$, $N$, $N_{\mathcal{R}}$, $W$, and $M$, where
$2W$ = Width of sample interval, and with
$M+1$ sample points

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$z_0 = N + 1/2 - W(1+i)$
$h = 2W(1+i)/M$

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# Quadrature Formula for $I_0(s)$

$$I_0(s) = \sum_{n=1}^{N} n^{-s} + h \sum_{m=0}^{M} f(s, z_0 + mh)$$

$$- \sum_{n=N_{\mathcal{L}}}^{N} H((N + 1/2 - n)/h)\, n^{-s}$$

$$+ \sum_{n=N+1}^{N_{\mathcal{R}}} H((n - N - 1/2)/h)\, n^{-s}$$

$$+ O(e^{-2\pi W^2 + \epsilon}) + O(e^{-\pi K M/W})$$

where

$$f(s, z) = \frac{\exp(i\pi z^2)\ z^{-s}}{e^{i\pi z} - e^{-i\pi z}} \quad \text{and}$$

$$K = \min(5/4 + N - N_{\mathcal{L}}, 1/4 + N_{\mathcal{R}} - N).$$



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# Computing $\zeta(s)$ to $28$ decimal places

$$
\begin{aligned}
s &= 1/2 + 10^5 i \\
N &= 126 \\
N_{\mathcal{L}} &= 121 \\
N_{\mathcal{R}} &= 131 \\
W &= 2.5 \\
M &= 50
\end{aligned}
$$

Computation time using quadrature was $0.31$ sec. in GP/PARI on a 300MHz Ultrasparc — vs 204 sec. (18 sec. after first computation) using PARI's built-in routine based on Euler-Maclaurin summation.



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# Computing $\zeta(s)$ to $1000$ decimal places

$$\begin{aligned} s &= 1/2 + 10^5 i \\ N &= 126 \\ N_{\mathcal{L}} &= 100 \\ N_{\mathcal{R}} &= 152 \\ W &= 13 \\ M &= 1500 \end{aligned}$$

Computation time using quadrature was $25$ minutes in GP/PARI on 300MHz Ultrasparc.

# Things to do

- Get explicit, tight, error bounds.
- Do a more careful complexity analysis.
- Generalize to other zeta functions and $L$-functions.

# Literature

- For a good survey of methods for computing $\zeta(s)$, see the preprint [BBC].
- For a careful error analysis of the Euler-Maclaurin formula for $\zeta(s)$, see [CO92].
- For a careful error analysis of the Riemann-Siegel formula, when $\sigma = 1/2$, see [Gab79].
- Methods for efficient computation of $\zeta(\sigma + it)$ at many equally spaced values of $t$ are described in [OS88].
- For "smoothed" versions of the Riemann-Siegel formula, with better accuracy than the classical version, see [BK92].
- For further discussion of quadrature of saddle point integrals, see [Tem77].



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