

NOTES ON NEWMAN'S "FIRST PROOF" OF THE PRIME NUMBER THEOREM

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ABSTRACT. This note attempts to clear up some remaining points in the "first proof" of the prime number theorem from *Analytic Number Theory* by Donald J. Newman [New98].

These notes assume that the reader is familiar with the background material that I presented in lectures for the [SFU Number Theory Study Group](#), November, 2001.

In the following discussion $f(x) \ll g(x)$ will mean $|f(x)| \leq C|g(x)|$ for some $C > 0$, in other words $f(x) = O(g(x))$. This notation differs from Newman, who uses a convention where $C = 1$.

Picking up the proof near the middle of page 71, we have reduced the task of showing $\pi(N) \sim N/\ln(N)$ to the task of showing that the average value of the coefficients of $(-\zeta'(s)/\zeta(s) - \zeta(s))$ approaches 0. Recall that

$$\begin{aligned}
 (1) \quad & -\frac{\zeta'(s)}{\zeta(s)} - \zeta(s) = \frac{1}{\zeta(s)} (-\zeta'(s) - \zeta^2(s)) \\
 & = \left(\sum_{n \geq 1} \mu(n)n^{-s} \right) \left(\sum_{n \geq 1} (\ln(n) - d(n))n^{-s} \right),
 \end{aligned}$$

where $d(n)$ counts the number of divisors of n .

Now, the average of the first N terms of (1) is

$$\begin{aligned}
 (2) \quad & \frac{1}{N} \sum_{ab \leq N} \mu(a)(\ln(b) - d(b)) \\
 & = \frac{1}{N} \sum_{ab \leq N} \mu(a)(\ln(b) - d(b) + 2\gamma) - \frac{2\gamma}{N} \sum_{ab \leq N} \mu(a) \\
 & = -\frac{2\gamma}{N} + \frac{1}{N} \sum_{ab \leq N} \mu(a)(\ln(b) - d(b) + 2\gamma),
 \end{aligned}$$

where we have used the fact that $\sum_{a|n} \mu(a) = \begin{cases} 1 & n = 1 \\ 0 & \text{otherwise.} \end{cases}$

In Equation (2), Newman picks γ to be *the constant for which*

$$(3) \quad \sum_{b=1}^K (\ln(b) - d(b) + 2\gamma) = O(K^{1/2}).$$

I'm almost certain that γ is “Euler’s constant”, $\gamma \approx 0.5772 \dots$. In any case, most of what’s needed to show that there is such a constant can be found in the discussion of “the average order of $d(n)$ ” in §3.5 of *Introduction to Analytic Number Theory* [Apo76].

Note that it’s a bit surprising that we can show that the expression (2) goes to zero as $N \rightarrow \infty$, since there are $\gg N \ln(N)$ nonzero terms in the sum over $ab \leq N$. The fact that this sum is so small depends on cancellation between terms, so we have to be careful not to take absolute values too soon when we bound this sum.

In the following discussion it’s convenient to define

$$M(x) := \sum_{n \leq x} \mu(n).$$

Recall that we’ve already shown that $M(N) = o(N)$ as $N \rightarrow \infty$. Following Newman, we can quantify this by writing

$$(4) \quad \left| \frac{1}{N} M(N) \right| \leq \delta(N),$$

where $\delta(N) \rightarrow 0$ as $N \rightarrow \infty$.

We will now use a variation on the idea behind “Dirichlet’s hyperbola method” to bound the sum of Equation (2). Given N , fix w with $1 \leq w \leq N$, and break the

sum that we want to bound into two disjoint subsums:

$$\begin{aligned}
 & \sum_{ab \leq N} \mu(a)(\ln(b) - d(b) + 2\gamma) \\
 &= \sum_{a \leq N/w} \mu(a) \sum_{b \leq N/a} (\ln(b) - d(b) + 2\gamma) \\
 & \quad + \sum_{b < w} (\ln(b) - d(b) + 2\gamma) \sum_{N/w < a \leq N/b} \mu(a).
 \end{aligned}$$

(To understand this breakdown, you might want to draw a sketch in the a - b plane of the region bounded by the hyperbola $ab = N$ and of the regions corresponding to each subsum.)

We can bound the first subsum using the fact that $|\mu(n)| \leq 1$ and the bound (3).

These yield

$$\begin{aligned}
 & \left| \sum_{a \leq N/w} \mu(a) \sum_{b \leq N/a} (\ln(b) - d(b) + 2\gamma) \right| \\
 & \leq \sum_{a \leq N/w} \left| \sum_{b \leq N/a} (\ln(b) - d(b) + 2\gamma) \right| \\
 (5) \quad & = \sum_{a \leq N/w} O\left(\sqrt{N/a}\right) \ll N/\sqrt{w}.
 \end{aligned}$$

To bound the second subsum, begin with the bound (4) to find

$$\begin{aligned}
 & \left| \sum_{N/w < a \leq N/b} \mu(a) \right| = |M(N/b) - M(N/w)| \\
 & \leq |M(N/b)| + |M(N/w)| \ll \frac{N}{b} \delta(N/w).
 \end{aligned}$$

We now combine this with the bound $\ln(b) - d(b) + 2\gamma \ll b$ (the trivial bound $d(b) \ll b$ dominates everything else) to find that

$$(6) \quad \sum_{b < w} (\ln(b) - d(b) + 2\gamma) \sum_{N/w < a \leq N/b} \mu(a) \ll \sum_{b < w} b \frac{N}{b} \delta(N/w) \ll Nw \delta(N/w).$$

Now we want to define w to be a function of N so that $w(N) \rightarrow \infty$ as $N \rightarrow \infty$, and also so that

$$(7) \quad w(N) \delta(N/w(N)) \rightarrow 0$$

as $N \rightarrow \infty$. Without loss of generality we can assume that $\delta(x)$ is a continuous function of x , and that $\delta(y) \leq \delta(x)$ whenever $y \geq x$. Furthermore, note that $\delta(x) > 0$ for $x < \infty$, since $\mu(n) = \pm 1$ for an infinity of n . With these observations, we see that a suitable $w(N)$ comes by defining N_m to be the least solution to $\delta(N_m/m) \leq \delta(1)/m^2$. Then let $w(N) = m$ for $N_m \leq N < N_{m+1}$. Note that $N_1 = 1$, and we see that N_m is an unbounded increasing sequence by the remarks above. Thus $w(N) \rightarrow \infty$, and the condition (7) easily follows from our definition.

With this choice of $w(N)$, applying the bounds (5) and (6) to Equation (2), we find that

$$\begin{aligned} \frac{1}{N} \sum_{ab \leq N} \mu(a)(\ln(b) - d(b)) &\ll \frac{1}{N} + \frac{1}{\sqrt{w(N)}} + w(N) \delta(N/w(N)) \\ &= o(1) \quad \text{as } N \rightarrow \infty. \end{aligned}$$

This is exactly what we needed to show that $\sum_{n \leq N} \Lambda(n) = N + o(N)$, and the Prime Number Theorem follows. In more detail, use

$$\Lambda(n) = \begin{cases} \ln(p) & n = p^m, p \text{ prime and } m \geq 1 \\ 0 & \text{otherwise,} \end{cases}$$

and then use integration by parts (or summation by parts) to relate $\pi(N)$ to $\sum_{n \leq N} \Lambda(n)$. (You'll also have to show that prime powers p^m with $m \geq 2$ don't matter much.)

REFERENCES

- [Apo76] Tom M. Apostol, *Introduction to analytic number theory*, Springer-Verlag, New York, 1976, Undergraduate Texts in Mathematics, [MR 55:7892](#).

[New98] Donald J. Newman, *Analytic number theory*, Springer-Verlag, New York, 1998,
[MR 98m:11001](#).

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