An Asymptotic Expansion of Ramanujan

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On page 324 of Ramanujan's second notebook we find

$$\begin{aligned} 2\sum_{n\geq 0}(-1)^n\left(\frac{1-t}{1+t}\right)^{n(n+1)} \\ \sim\ 1+t+t^2+2t^3+5t^4+17t^5\cdots \end{aligned}$$

as t tends to $0+$.

To see that the leading coefficient might be correct, let $t=0$ and we have

$$\begin{aligned} 2\sum_{n\geq 0}(-1)^n &= 2(1/2) \\ &= 1 \end{aligned}$$

Notation

We want to study the coefficients a_m in the asymptotic expansion of

$$F(q) = 2 \sum_{n \geq 0} (-1)^n q^{n(n+1)}$$
$$\sim \sum_{m \geq 0} a_m t^m$$

where $q = (1-t)/(1+t)$. The first series converges for $|q| < 1$, and thus for $\Re(t) > 0$. $F(q)$ is one of L. J. Rogers' false theta functions.

For technical reasons, it is easier to work with

$$s = t/(1+t)$$

so

$$t = s/(1-s)$$

and

$$q = \frac{1-t}{1+t} = 1 - 2s$$

The corresponding region of convergence is $|s - \frac{1}{2}| < \frac{1}{2}$.

Writing the expansion in s as

$$F(q) \sim \sum_{m \geq 0} b_m \, s^m$$

and using

$$\begin{aligned} s^j &= \left(\frac{t}{1+t}\right)^j \\ &= \sum_{k \geq 0} \binom{-j}{k} t^{j+k} \end{aligned}$$

the expansion in t is given by

$$\begin{aligned} a_m &= \sum_{j+k=m} \binom{-j}{k} b_j \\ &= \sum_{j=0}^{m} \binom{-j}{m-j} b_j \end{aligned}$$

Thus $F(q) \in \mathbb{Z}[[s]] \Rightarrow F(q) \in \mathbb{Z}[[t]]$.

m	a_m	b_m
0	1	1
1	1	1
2	1	2
3	2	5
4	5	15
5	17	54
6	72	233
7	367	1191
8	2179	7080
9	14750	48025
10	112023	365761
11	942879	3087824
12	8708912	28604041
13	87563937	288378765
14	951933849	3142778610
15	11125383714	36811949617

A Formula from the "Lost" Notebook

$$\sum_{n\geq 0}(-1)^n q^{n(n+1)} = \sum_{n\geq 0}\frac{(q;q^2)_n^2\, q^n}{(-q;q)_{2n+1}}$$

where

$$(a;q)_n = \prod_{j=0}^{n-1}(1-aq^j)$$

(A proof appears in *Ramanujan's "Lost" Notebook. I. Partial θ-Functions*, by George E. Andrews, Advances in Mathematics Vol. 41. No. 2, August 1981.)

Writing each term as

$$T_n = \frac{(q;q^2)_n^2\, q^n}{(-q;q)_{2n+1}}$$

we see that $T_n = O(s^{2n})$, $(s \to 0)$, since

$$\begin{aligned} (q;q^2)_n &= \prod_{j=0}^{n-1} (1 - q^{2j+1}) \\ &= \prod_{j=0}^{n-1} O(s) \\ &= O(s^n) \end{aligned}$$

while the denominator in T_n is bounded away from 0, and $q = O(1)$ as $s \to 0$.

Further, for $s \in (0, 1/2)$, so $q \in (0, 1)$, we have

$$\begin{aligned} T_{n+1} &= q\frac{(1-q^{2n+1})^2}{(1+q^{2n+2})(1+q^{2n+3})}T_n \\ &\leq q\,T_n \\ T_{n+k} &\leq q^k T_n \end{aligned}$$

thus

$$\begin{aligned} \sum_{k\geq 0} T_{n+k} &\leq T_n \sum_{k\geq 0} q^k \\ &= T_n/(2s) \\ &= O(s^{2n-1}) \end{aligned}$$

We can refine this error bound to get

$$\begin{aligned} \sum_{k\geq 0} T_{n+k} &= T_n + \sum_{k\geq 0} T_{n+1+k} \\ &= O(s^{2n}) + O(s^{2n+1}) \\ &= O(s^{2n}) \end{aligned}$$

This gives

$$\begin{aligned} F(q) &= 2 \sum_{n \geq 0} (-1)^n q^{n(n+1)} \\ &= \sum_{n=0}^{N-1} 2\, T_n + O(s^{2N}) \end{aligned}$$

Expanding each T_n as a power series in s (or t) through the $2N-1$'th coefficient gives the first $2N$ coefficients for $F(q)$. $N = 3$ gives the six coefficients computed by Ramanujan.

Integrality of the coefficients

To show that $F(q) \in \mathbb{Z}[[t]]$ it suffices to show that $2\,T_n \in \mathbb{Z}[[s]]$ for every n.

For $k \in \mathbb{Z}$ we have

$$q^k \in 1 + 2s\mathbb{Z}[[s]]$$

so

$$\begin{aligned} 1 - q^k &\in 2s\mathbb{Z}[[s]] \subset 2\mathbb{Z}[[s]] \\ 1 + q^k &\in 2 + 2s\mathbb{Z}[[s]] \end{aligned}$$

Writing

$$\begin{aligned} 2\,T_n &= \frac{2(q;q^2)_n^2}{(-q;q)_{2n+1}} q^n \\ &= \frac{2\prod_{j=0}^{n-1}(1-q^{2j+1})^2}{\prod_{j=0}^{2n}(1+q^{j+1})} q^n \end{aligned}$$

the numerator of the ratio contains $2n+1$ factors of the form $2\mathbb{Z}[[s]]$, while the denominator contains $2n+1$ factors of the form $2+2s\mathbb{Z}[[s]]$. The desired result follows since $q^n \in \mathbb{Z}[[s]]$, and since

$$\frac{2\mathbb{Z}[[s]]}{2+2s\mathbb{Z}[[s]]} = \frac{\mathbb{Z}[[s]]}{1+s\mathbb{Z}[[s]]} \subseteq \mathbb{Z}[[s]]$$

Areas for further work

- Can we give an asymptotic estimate for the coefficients?
- Study the asymptotic behavior of $F(q)$ as q approaches other roots of unity.