Aequationes Mathematicae 40 (1990) 111–135
University of Waterloo

0001-9054/90/030111-25$1.50 + 0.20/0

Survey Paper

A survey of Sylvester's problem and its generalizations

P. Borwein and W. O. J. Moser

Summary. Let P be a finite set of three or more noncollinear points in the plane. A line which contains two or more points of P is called a *connecting* line (determined by P), and we call a connecting line *ordinary* if it contains precisely two points of P. Almost a century ago, Sylvester posed the disarmingly simple question: *Must every set P determine at least one ordinary line?* No solution was offered at that time and the problem seemed to have been forgotten. Forty years later it was independently rediscovered by Erdös, and solved by Gallai. In 1943 Erdös proposed the problem in the American Mathematical Monthly, still unaware that it had been asked fifty years earlier, and the following year Gallai's solution appeared in print. Since then there has appeared a substantial literature on the problem and its generalizations.

In this survey we review, in the first two sections, Sylvester's problem and its generalization to higher dimension. Then we gather results about the connecting lines, that is, the lines containing two or more of the points. Following this we look at the generalization to finite collections of sets of points. Finally, the points will be colored and the search will be for monochromatic connecting lines.

1. Introduction

Let a finite set of points in the plane have the property that the line through any two of them passes through a third point of the set. Must all the points lie on one line? Almost a century ago Sylvester (1893) posed this disarmingly simple question. No solution was offered at that time and the problem seemed to have been forgotten. Forty years later it resurfaced as a conjecture by Erdös: *If a finite set of points in the plane are not all on one line then there is a line through exactly two of the points.* In a recent reminiscence Erdös (1982, p. 208) wrote: "*I expected it to be easy but to my great surprise and disappointment I could not find a proof. I told this problem to Gallai who very soon found an ingenious proof.*" In 1943 Erdös proposed the problem in the American Mathematical Monthly (Erdös 1943), still unaware that it had been asked fifty years earlier, and the following year Gallai's solution

AMS (1980) subject classification: Primary 51A20; Secondary 52A37.

Manuscript received October 17, 1989, and in final form, March 15, 1990.

appeared in print (Gallai 1944). Since then there has appeared a substantial literature on the problem and its generalizations, and two long-standing conjectures have recently been settled. But there are still unanswered questions, unsettled conjectures, and a survey paper at this time seems appropriate.

In the following two sections we review Sylvester's problem and its generalization to higher dimension. Then we will gather the properties of the connecting lines, that is, the lines containing two or more of the points. Following this we will look at the generalization to finite collections of sets of points. Finally, the points will be colored and the search will be for monochromatic connecting lines.

2. Sylvester's problem

The answer to Sylvester's question is negative in the complex projective plane and in some finite geometries (Coxeter 1948) so we restrict our attention to the ordinary real plane—Euclidean, affine, projective. (For interesting remarks and theorems in the "complex" case see L. M. Kelly (1986) and Boros, Füredi and Kelly (1989).) Let P be a finite set of three or more noncollinear points in the plane. A line which contains two or more points of P is called a *connecting* line (determined by P), and we call a connecting line *ordinary* if it contains precisely two points of P. In this terminology, Sylvester's theorem is: *Every set P determines at least one ordinary line.* Because Gallai's proof came first we give it here although it played no role in subsequent developments. Here then is the affine proof by Gallai (1944) (also in: de Bruijn and Erdös 1948; Hadwiger, Debrunner and Klee 1964, p. 57; Croft 1967). Choose any point $p_1 \in P$. If p_1 lies on an ordinary line we are done, so we may assume that p_1 lies on no ordinary line. Project p_1 to infinity and consider the set of connecting lines containing p_1. These lines are all parallel to each other, and each contains p_1 and at least two other points of P. Any connecting line not through p_1 forms an angle with the parallel lines; let s be a connecting line (not through p_1) which forms the smallest such angle (Figure 1a). Then s must be ordinary! For suppose s were to contain three (or more) points of P, say p_2, p_3, p_4 named so that p_3 is between p_2 and p_4 (Figure 1b). The connecting line through p_3

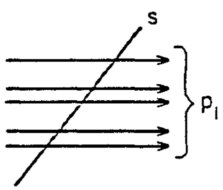

Figure 1a.

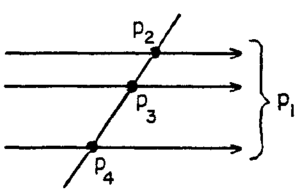

Figure 1b.

and p_1 (being not ordinary) would contain a third point of P, say p_5, and now either the line p_2p_5 or the line p_4p_5 would form a smaller angle with the parallel lines than does s.

L. M. Kelly provided a particularly simple, pleasing Euclidean proof (in Coxeter 1948, 1969, p. 65; Croft 1967) and for this reason we give it now. We have the set P of noncollinear points and the set S of connecting lines determined by P. Any point in P and any connecting line not through the point determine a perpendicular distance from the point to the line. The collection of all these distances is finite, because P and S are finite, so there is a smallest such distance. Let p^* (in P) and s^* (in S) be a nonincident pair which realizes this smallest distance. Then s^* is ordinary! For otherwise s^* would contain at least three points of P and two of them would have to lie on the same side of q, the foot of the perpendicular from p^* to s^*, as in Figure 2a, where the two points on the same side of q are p_1, p_2 with p_1 between q and p_2 ($p_1 = q$ is a possibility, as in Figure 2b). Now the distance from p_1 to the connecting line p^*p_2 would be smaller than the distance from p^* to s^*.

We give still one more proof, that of Steinberg (1944), because it turned out to be the springboard for what was to come. It is a projective proof; the basic properties of the r.p.p. (real projective plane) may be found in many monographs, e.g., Coxeter (1955, 1969). Here now is Steinberg's proof (also in Coxeter 1948, 1955, p. 30; Motzkin 1951). We have a finite set P of noncollinear points in the projective plane and the set S of connecting lines determined by P. Let p be any point of P. If p lies on an ordinary line we are done, so we may assume that p lies on no ordinary line. Let l be a line (in the plane) through p but not through any other point of P. The lines in S not through p meet l, in points $x_1, x_2, \ldots, x_k$ say, named in cyclic order so that one of the two segments determined by p and x_1 contains none of the points $x_2, x_3, \ldots, x_k$ within it (see Figure 3a). Let s be a line of S through x_1 (see Figure 3b). Then s must be ordinary! For otherwise there would be three or more points of P on s, say p_1, p_2, p_3, named so that p_1 and x_1 are separated by p_2 and p_3 (see Figure 3c). The connecting line through p and p_1 would have to contain a further point of P (remember, p lies on no ordinary line), say p_4, and then one of these two connecting lines p_2p_4, p_3p_4 would meet the "forbidden" segment.

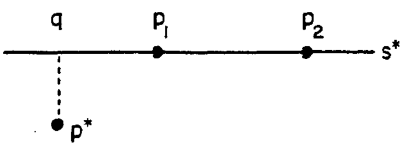

Figure 2a.

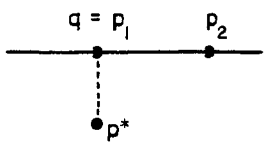

Figure 2b.

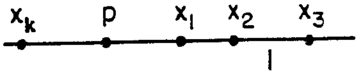

Figure 3a.

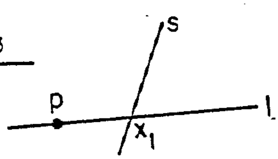

Figure 3b.

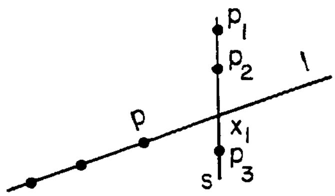

Figure 3c.

Proofs of Sylvester's theorem were also given by R. C. Buck (see Erdös 1944), A. Robinson (see Motzkin 1951), Lang (1955), Steenrod (see Erdös 1944; Chakerian 1970), Williams (1968) and probably others.

Knowing that every set P determines an ordinary line, a natural question to ask is whether every set P determines more than one ordinary line. Let $m(P)$ denote the number of ordinary lines determined by P, and set

$$m(n) = \min_{|P|=n} m(P) \tag{1}$$

($|X|$ denotes the number of elements in the set X); that is, $m(n)$ is the *least* number of ordinary lines determined by any set P of n noncollinear points in the plane. In this notation, Sylvester's theorem states that

$$m(n) \geqslant 1 \qquad \text{for } n \geqslant 3, \tag{2}$$

and attention turned to finding better inequalities. de Bruijn and Erdös (1948) observed that

> ... *it is not known whether* $\lim_{n\to\infty} m(n) = \infty$. *All we can prove is that*
> $$m(n) \geqslant 3 \qquad \text{for } n \geqslant 3. \tag{3}$$

A rather lengthy affine proof of (3) was given by Dirac (1951). It is interesting to note that an equivalent version of (3) had been proved earlier by Melchior (1940), and probably much earlier by others, though the connection with Sylvester's problem went unnoticed. Melchior studied configurations of lines in the r.p.p. To understand the connection between these configurations and the set of points we have been looking at, we apply the Principle of Duality (in the r.p.p.). For example, consider the following definitions and notations:

(a) P is a finite set of points not all on one line;
(b) a connecting line (determined by P) contains two or more points of P;
(c) $S = S(P)$ is the set of connecting lines determined by P;

(d) an i-line, $i \geqslant 2$, is a connecting line containing exactly i points of P;
(e) a 2-line is called ordinary;
(f) $t_i = t_i(P)$ denotes the number of i-lines determined by P;
(g) $|S| = \sum_{i \geqslant 2} t_i(P)$;
(h) $m(P) = t_2(P)$;
(i) $m(n) = \min_{|P|=n} m(P)$.

In the dual configuration we have correspondingly:

(a) L is a finite set of lines not all through one point;
(b) a point of intersection (determined by L) lies on two or more lines of L;
(c) V is the set of points of intersection determined by L;
(d) an i-point, $i \geqslant 2$, is a point lying on exactly i lines of L;
(e) a 2-point is called simple;
(f) $v_i = v_i(L)$ denotes the number of i-points determined by L;
(g) $|V| = \sum_{i \geqslant 2} v_i(L)$;
(h) $m'(L) = v_2(L)$;
(i) $m'(n) = \min_{|L|=n} m'(L)$.

Because of the duality, it follows that, if L is the dual of P, then $v_i(L) = t_i(P)$, $v_i(L) = m_i(P)$, $m'(n) = m(n)$, and so on. Now consider the set L. Its lines partition the r.p.p. into faces—regions whose vertices are in V and whose edges are segments of the lines in L. Let $V(L)$, $E(L)$ and $F(L)$ denote the number of vertices, edges and faces, respectively, in this partition (dissection) of the plane. These numbers satisfy the Euler–Poincaré formula

$$V(L) - E(L) + F(L) = 1. \tag{4}$$

If we let $f_i(L)$ denote the number of (the $F(L)$) faces having exactly i sides, then

$$V(L) = \sum_{i \geqslant 2} v_i(L), \quad F(L) = \sum_{i \geqslant 3} f_i(L), \quad 2E(L) = \sum_{i \geqslant 3} if_i(L) = 2 \sum_{i \geqslant 2} iv_i(L), \tag{5}$$

and substituting in (4) we find that

$$\begin{aligned} 3 &= 3V(L) - E(L) + 3F(L) - 2E(L) \\ &= 3 \sum_{i \geqslant 2} v_i(L) - \sum_{i \geqslant 2} iv_i(L) + 3 \sum_{i \geqslant 3} f_i(L) - \sum_{i \geqslant 3} if_i(L) \\ &= \sum_{i \geqslant 2} (3-i)v_i(L) + \sum_{i \geqslant 3} (3-i)f_i(L), \end{aligned}$$

or

$$v_2(L) = 3 + \sum_{i \geq 4} (i-3)v_i(L) + \sum_{i \geq 4} (i-3)f_i(L)$$

and hence

$$v_2(L) \geq 3 + \sum_{i \geq 4} (i-3)v_i(L). \tag{6}$$

Thus, any finite set of lines not all through one point determine at least three simple points of intersection (Melchior 1940); equivalently, any finite set of points not all on one line determine at least three ordinary lines!

The dual of (6) is

$$t_2(P) \geq 3 + \sum_{i \geq 4} (i-3)t_i(P) = 3 + t_4(P) + 2t_5(P) + 3t_6(P) + \cdots. \tag{7}$$

W. Moser (1957) used this inequality to prove that

$$m(P) > \frac{n+11}{6} \qquad \text{if } |P| = n \text{ is even,} \tag{8}$$

by the following argument. The number of points in P each incident with a k-line for at least one $k \neq 3$ is at most

$$\begin{aligned} &2t_2(P) + 4t_4(P) + 5t_5(P) + \cdots \\ &\quad \leq 2t_2(P) + 4(t_4(P) + 2t_5(P) + 3t_6(P) + \cdots) \end{aligned}$$

(and from (7))

$$\leq 2t_2(P) + 4(t_2(P) - 3) = 6t_2(P) - 12.$$

If $t_2(P) \leq (n+11)/6$ then $6t_2(P) - 12 \leq n - 1$ and consequently at least one point of P lies solely on 3-lines, implying that n is odd (Kelly and Moser 1958). In other words, if n is even then $t_2(P) > (n+11)/6$.

It is appropriate at this point to mention the connection between configurations of points (and their connecting lines) in the r.p.p. and the class of polyhedra known as zonohedra. First recall that a polygon in the Euclidean plane is said to have central symmetry, or to be centrally symmetric, if there is a point C (called the center of the polygon) which bisects every chord of the polygon through it. Figure 4 exhibits a centrally symmetric convex 4-gon, 6-gon and 8-gon.

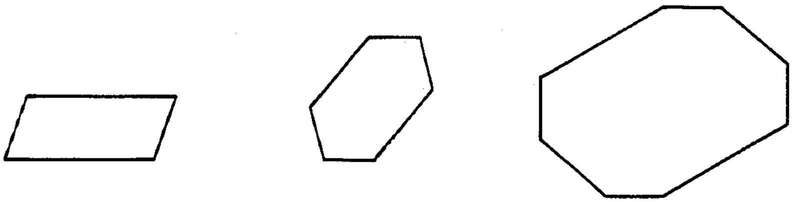

Figure 4.

Similarly, in E^3 (Euclidean 3-space) a polyhedron is centrally symmetric with center C if every chord through C is bisected at C. A convex polyhedron all of whose faces have central symmetry is called a zonohedron (Coxeter 1968, Chapter 4; Moser 1961). A. D. Alexandroff proved that a zonohedron must have 3-dimensional central symmetry (see Burckhardt 1940). Let Z be a zonohedron. Choose an edge of Z and notice that it determines a set of parallel edges, and also a "zone" of faces, each containing two sides in the set of parallel edges. Let O be a fixed point in space. Corresponding to a zone, place a line through O parallel to the edges of the associated set of parallel edges. In this way we are led from a zonohedron, which has n zones say (i.e., its edges fall into n sets of parallel edges), to a "star" of n lines through O (Coxeter 1973, p. 28; photographs of some zonohedra are shown in Plate II, p. 33). This star determines planes, each containing two or more lines of the star. The star and these planes intersect the plane at infinity in a set P of n points and their connecting lines. The correspondence between Z, the star and P is indicated in the following table.

Zonohedron Z in E^3	Star of lines through O	Set P of points in the rpp
zone of faces	line	point
n zones	n lines	n points
a pair of opposite faces	a plane containing 2 or more of the lines	a connecting line
a pair of opposite $2i$-gons	a plane containing exactly i lines	an i-line
a pair of opposite parallelograms	a plane containing exactly 2 lines	an ordinary line

L. Moser observed long ago that a zonohedron must have at least three pairs of (opposite) parallelogram faces. (The proof of this is essentially the same as the proof of (6).) That this is equivalent to (3) also seems to have gone unnoticed.

As Motzkin observed, Steinberg's proof of (2) shows that in the dissection of the r.p.p. by the ordinary lines, each of the polygonal regions contains at most one point of P in its interior. Since there at most $\binom{n(P)}{2} + 1$ regions (easily proved by induction), and at least $|P| - 2m(P)$ points which lie on no ordinary line, it follows that

$$|P| - 2m(P) \leqslant 1 + \frac{1}{2} m(P)(m(P) - 1),$$

and hence

$$m(n) > \sqrt{2n} - 2$$

(Motzkin 1951). W. Moser (1957) took a small step forward by proving $m(n) > \sqrt{5n/2}$, and Kelly and Moser (1958) took a big step by proving

$$m(n) \geqslant \frac{3n}{7}, \qquad n \geqslant 3. \tag{9}$$

The essence of their proof goes as follows. Consider a point p in P. The lines of S not through p dissect the plane into polygonal regions. The point p lies inside one of these regions, called the *residence* of p, and the connecting lines containing the edges of the residence of p are (called) the *neighbors* of p. Suppose that s (in S) is a neighbor of three points p_1, p_2, p_3 in P. It is easy to see that the only possible locations for points of P on s are at the intersections of s with the three connecting lines determined by p_1, p_2, p_3, i.e., at the "open" circles in Figure 5a. Consequently, if 4 points p_1, p_2, p_3, p_4 of P have a common neighbor s (in S) then the only possible locations for the points of P on s are at the diagonal points—the "open" circles in Figure 5b—and s must be *ordinary*. Furthermore, if a point p in P lies on precisely one ordinary line then it must have at least two ordinary neighbors. (The faulty proof of this by Kelly and Moser was corrected by Dirac (1959).) It follows

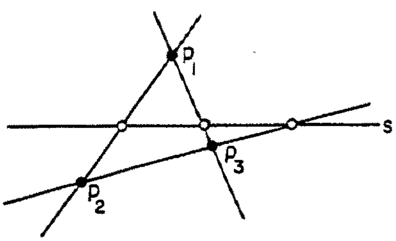

Figure 5a.

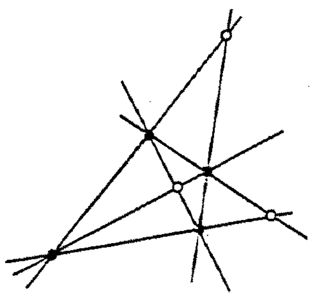

Figure 5b.

that if there are k points of P each lying on precisely two ordinary lines then

$$m(P) \geqslant k, \qquad m(P) \geqslant \frac{3(n-k)+2k}{6}$$

and consequently $m(P) \geqslant 7$.

Of course for any set P with n points,

$$m(n) \leqslant m(P),$$

so upper bounds on $m(n)$ are obtained by looking at particular sets P with $|P| = n$. Figure 6 (Kelly and Moser 1958) exhibits a configuration of 7 points which determine 3 ordinary lines, showing that $m(7) \leqslant 3$. Since $m(7) \geqslant 3$ ((9) with $n = 7$) we have $m(7) = 3$.

Let $A\,B\,C\,D\,E\,C'\,D'\,E'$ be the vertices of two regular pentagons (in the Euclidean plane) which have the common edge AB (see Figure 7). These 8 points together with M (the midpoint of segment AB) and $I\,J\,K$ and L (the points at infinity in the directions CD, MD, ED and AB respectively) are a set of 13 points determining 6 ordinary lines (dotted in the diagram). This configuration, due to McKee (Crowe and McKee 1968), shows that $m(13) \leqslant 6$, and hence $m(13) = 6$.

The following configurations are due to Motzkin and Böröczky (see Crowe and McKee 1968), and show that

$$m(n) \leqslant \begin{cases} n/2 & \text{if } n \text{ is even,} \\ 3(n-1)/4 & \text{if } n \equiv 1 \pmod 4, \\ 3(n-3)/4 & \text{if } n \equiv 3 \pmod 4. \end{cases} \qquad (10)$$

If $n = 2k$ then, to the set of k vertices of a regular k-gon (in the Euclidean plane),

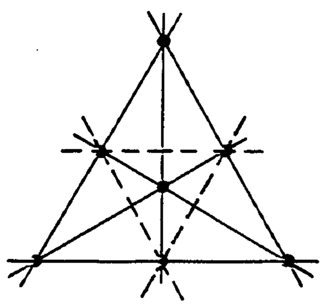

Figure 6.

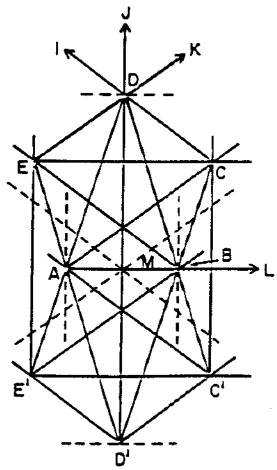

Figure 7.

add the k points at infinity determined by the lines through pairs of vertices (Figure 8 shows the configuration when $k = 6$). This set of $2k$ points in the real projective plane determines exactly k ordinary lines. If $n = 4k + 1$ then, to the configuration (above) for $4k$ points, add the center C of the $2k$-gon. We now have a set of $4k + 1$ points which determine exactly $3k$ ordinary lines: one through each of the vertices of the $2k$-gon and k ordinary lines through C. In the case $n = 4k + 3$, delete from the configuration for $4k + 4$ points a point E on l_∞ in a direction *not* determined by an edge of the polygon. (Figure 9 shows the configuration for $k = 3$, i.e., $n = 11$.) This configuration has one ordinary line through each of $2k$ vertices of the polygon—the two vertices which fail to lie on ordinary lines are opposite and their

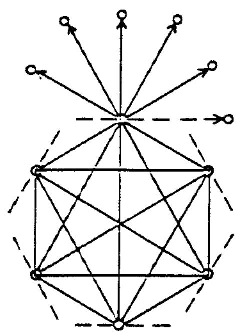

Figure 8. $m(12) \leqslant 6$.

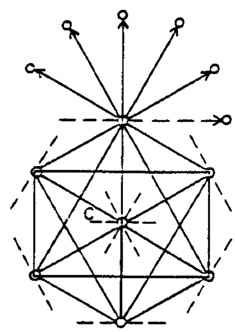

Figure 9. $m(13) \leqslant 9$.

join is perpendicular to the direction of E—and another k parallel ordinary lines in the direction of E.

Using these lower and upper bounds ((9 and (10) respectively) and the more or less obvious

$$\binom{n}{2} = \sum_{i=2} \binom{i}{2} t_i(P)$$

(Kelly and Moser 1958), Crowe and McKee (1968) established (exact) values of $m(n)$ for small n. Their list was extended by Brakke (1972):

$n =$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$m(n) =$	3	3	4	3	3	4	6	5	6	6	6	7		8		9				11.

Dirac (1951) conjectured that $m(n) \geqslant n/2$ for $n \neq 7$. Since $m(13) = 6$, the correct conjecture is

$$m(n) \geqslant \frac{n}{2} \qquad \text{for } n \neq 7, 13, \tag{11}$$

and this was repeated by Kelly and Moser (1958).

Rottenberg (1973) investigated the consequences of assuming that a set P determines fewer than $n/2$ ordinary lines. It is easy to see that such a set must contain a point which has no ordinary neighbors and is incident with precisely two ordinary lines, and he made a detailed analysis (Rottenberg 1982) of a set which contained such "singular" points. But he was unable to close the gap between bound (9) and conjecture (11).

A lengthy (100 pages!!) "proof" of (11) has been given by Hansen (1981) in his Doctoral dissertation. We have set "proof" in quotation marks because, to the best of our knowledge, nobody has been able to completely follow and verify the arguments in Hansen's proof.

A natural generalization of the dual version of Sylvester's problem is to consider an arrangement of pseudolines, a finite family of simple closed curves in the real projective plane each two of which have exactly one point in common at which point they cross. For example, Kelly and Rottenberg (1972) proved that such an arrangement has at least $3n/7$ simple vertices. For related results see Meyer (1974) and Watson (1980). In another direction Woodall (1969) obtained set-theoretic Sylvester-type results.

3. Higher dimensions

On each of two skew lines in 3-space place three points. This configuration (Motzkin 1951) is a counterexample to: *If a finite set of points in 3-space is not all on one plane then there is a plane spanned by and containing precisely three of the points.*

The correct generalization to a finite set of points in d-dimensional space ($d \geqslant 3$) is as follows. Let P be a finite set of points spanning d-dimensional (ordered projective) space. A $(d-1)$-flat F spanned by points of P is *ordinary* if all but one of the points in $F \cap P$ are in a $(d-2)$-flat. Let $\lambda_d(P)$ denote the number of ordinary flats determined by P, and $\lambda_d(n)$ the minimum of $\lambda_d(P)$ of all sets with $|P| = n$. Thus Hansen's result is

$$\lambda_2(n) \geqslant \frac{n}{2} \qquad \text{for } n \neq 7, 13.$$

Known results are as follows:

$\lambda_3(n) \geqslant 4$	Motzkin (1951)
$\lambda_d(n) \geqslant 1$ for $d \geqslant 3$	Hansen (1965)
$\lambda_3(n) \geqslant \frac{3n}{11}$	Bonnice and Kelly (1971)
$\lambda_3(n) \geqslant \frac{2n}{5}$	Hansen (1980)
$\lambda_3(n) \geqslant \frac{3n}{7}$ if P contains no elementary plane (i.e., a plane with exactly 3 non-collinear points on it). Conjecture in this case: $\lambda_3(P) \geqslant n$.	Bonnice and Kelly (1971)
$\lambda_3(n) \geqslant \frac{6n}{11}$ if P contains no elementary plane	Hansen (1980)
$\lambda_3(n) \geqslant \frac{n}{3}$ if P is not a subset of the union of two planes.	Rottenberg (1973)

In Section 2 we noted that in the plane a connecting line has at most 4 neighbors. It is easy to deduce that if it has 4 neighbors then it must be ordinary (Kelly and Moser 1958, p. 213 Remark). The corresponding result in three dimensions was obtained by Bonnice and Kelly (1971) and Rottenberg (1971), and in d dimensions, $d \geqslant 2$, by Rottenberg (1981): a connecting hyperplane has at most 2^d neighbors, and if it has 2^d neighbors then it must be elementary, that is, it contains and is spanned by precisely $d+1$ points of P.

Bonnice and Edelstein (1967) generalized elementary and ordinary flats determined by a set P spanning d-dimensional projective space to: α is a k-flat $(\bmod\, m)$, $m = 0, 1, \ldots, k-1$, if α is spanned by $P \cap \alpha$ and an m-flat μ exists such that all but $k-m$ points of $P \cap \alpha$ lie on μ. Note that a flat is elementary if and only if it is a k-flat (mod 0); it is an ordinary hyperplane (in projective $(k+1)$-space) if and only if it is a k-flat $(\bmod (k-1))$. They obtained a number of interesting results including a generalization of the following theorem of Motzkin (1951): Given a set P spanning 3-dimensional space, there is a plane π and two lines l_1 and l_2 in π such that $l_1 \cap l_2 = \{p\}$, $P \cap l_1$ spans l_1, $P \cap l_2$ spans l_2, and all the points of $P \cap \pi$ are contained in $l_1 \cup l_2$.

4. Connecting lines

Now we return to the set P in the plane. As before, a connecting line which contains exactly i points of P is called an i-line, and $t_i(P)$ denotes the number of i-lines in the set of connecting lines determined by P. We always assume $|P| = n$ and $t_n(P) = 0$ even when we do not explicitly say so.

At the same time that Erdös (1943) revived Sylverster's problem, he also asked whether

$$t(P) := \sum_{i \geqslant 2} t_i(P) \geqslant n,$$

i.e., are at least n connecting lines determined? This was proved by Erdös, Hanani (1951), Steinberg, Buck, Grünwald (=Gallai) and Steenrod (see Erdös 1944) and by others.

Kelly and Moser (1958) proved that

$$t(P) \geqslant kn - \frac{1}{2}(3k+2)(k-1) \qquad \text{if } \begin{cases} n \geqslant \frac{1}{2}\{3(3k-2)^2 + 3k - 1\}, \\ t_i(P) = 0 \text{ for } i > n-k, \end{cases}$$

i.e., if k is small compared to n and at most $n-k$ points of P are on a line then P determines almost kn connecting lines. This was a small step towards settling Erdös' conjecture: *There exists an absolute constant c independent of k and n such that if*

$0 \leqslant k \leqslant 2$ *and* $t_i(P) = 0$ *for* $i > n - k$ *then*

$$ckn < t(P) < 1 + kn.$$

(The upper bound is trivial.) This conjecture has recently been proved by Beck (1983). What is the best (or a "good") value of c? Is, for example, the conjecture true with $c = \frac{1}{6}$?

Szemerédi and Trotter (1983a, b, c) proved that there exists a constant c such that, if $k \leqslant \sqrt{n}$, then

$$\sum_{i \geqslant k} t_i(P) < c \frac{n^2}{k^3},$$

while Beck (1983) obtained independently a slightly weaker bound. That this estimate is best possible can be seen by considering the $\sqrt{n} \times \sqrt{n}$ lattice. This settles a conjecture of Erdös and Purdy (see Erdös 1985). In particular, there exists a constant c such that $\sum_{i \geqslant \sqrt{n}} t_i(P) \leqslant c\sqrt{n}$. Sah (1987) has constructed a set P, $|P| = n$, for which $\sum_{i \geqslant \sqrt{n}} t_i(P) \leqslant 3\sqrt{n}$. Consider the function (of the constant $c \geqslant 0$)

$$f_c(n) = \max_{|P| = n} \sum_{i \geqslant c\sqrt{n}} t_i(P).$$

Erdös asks: Is $f_c(n)/\sqrt{n}$ discontinuous at $c = 1$? Is $F_{(1+c)\sqrt{n}}(n)$ for all $c > 0$? Perhaps one could at least show that $F_{(1+c)\sqrt{n}}(n) < 2\sqrt{n}$.

Let

$$t_k(n) = \max\{t_k(P) \colon |P| = n\}, \qquad k \geqslant 3.$$

Erdös (1984a, b) asked for estimates of, or bounds on, $t_k(n)$. From the obvious identity

$$\sum_{i \geqslant 2} \binom{i}{2} t_i(P) = \binom{n}{2}$$

(both sides count the number of pairs of points in P) we have the trivial bound

$$t_k(n) \leqslant \binom{n}{2} \Big/ \binom{k}{2}, \qquad k \leqslant 3.$$

The numbers $t_3(n)$ have been studied for more than 150 years. The trivial bound

$$t_3(n) \leqslant \frac{1}{3}\binom{n}{2}$$

can be slightly improved, for

$$t_2(P) + 3t_3(P) \leqslant \binom{n}{2}$$

and

$$t_2(P) \geqslant \frac{n}{2}, \qquad n \neq 7,\ 13,$$

so

$$t_3(P) \leqslant \frac{1}{3}\left(\binom{n}{2} - \frac{n}{2}\right) = \frac{1}{6}n(n-2), \qquad n \neq 7,\ 13.$$

The best lower bound known is

$$1 + \left[\frac{1}{6}n(n-3)\right] \leqslant t_3(n),$$

proved by Burr, Grünbaum and Sloane (1974), who give an extensive bibliography and conjecture equality except for a few small n.

For $k \geqslant 4$, the trivial bound is

$$t_k(n) < \frac{1}{k(k-1)}n^2.$$

Erdös and Croft proved that if $k \geqslant 4$ there is a constant c_k (which depends on k) such that

$$c_k n^2 < t_k(n), \qquad n > n_0$$

(for a proof see Grünbaum 1972), but "good" values for c_k are not known.

Erdös also asked: If no connecting line contains more than k points of P, how many k-lines can P determine? That is, estimate or find bounds on

$$t'_k(n) = \max\{t_k(P): |P| = n,\ t_i(P) = 0 \text{ for } i > k\},$$

Of course $t'_k(n) \leqslant t_k(n)$. Kárteszi (1963) showed

$$t'_k(n) \geqslant c_k n \log n$$

(so $t'_k(n)/n \to \infty$ for any fixed k) and Grünbaum improved this to

$$t'_k(n) \geqslant cn^{1+\frac{1}{k-2}}$$

(see Erdös and Purdy 1976, p. 307). Erdös (1980, p. 49; 1984c) conjectures that, for any $k \geqslant 4$,

$$t'_k(n) = o(n^2)$$

(and offers \$100 for a proof or disproof), though he "$\cdots$ *cannot even prove that for any* $\epsilon > 0$

$$t'_4 \leqslant \frac{1}{12}(1-\epsilon)n^2 \qquad \text{when } n > n_0\text{''}.$$

Some of these results (for $t_k(n)$ and $t'_k(n)$) can be obtained in the dual formulation of a set of lines and their points of intersection (see Grünbaum 1976).

Grünbaum asked: What are the possible values that $t(P)$ can take? Clearly $t(P) \leqslant \binom{n}{2}$ and $t(P)$ can never equal $\binom{n}{2} - 1$ nor $\binom{n}{2} - 3$. Erdös (1972, 1973, p.18) showed that, for every q with $cn^{3/2} < q < \binom{n}{2} < 3$, there are sets P for which $t(P) = q$. Grünbaum conjectures (see Guy 1971) that the conclusion holds when $10 \leqslant 2n - 4 \leqslant q$, and Guy (1971) gives a counter conjecture. Recently Grünbaum's question has been answered by Salamon and Erdös (1988); they have characterized, for large n, the possible values for the number of connecting lines determined by n points.

Dirac (1951) showed that there is a point in P which lies on more than $\sqrt{n}$ connecting lines and conjectured that there is a point in P which lies on more than cn connecting lines (c a constant independent of n). This has been proved by Szemerédi and Trotter (1983a, c) and also by Beck (1983)—with c small—but still unsettled is Dirac's stronger conjecture: *There is a point in P which lies on at least* $[n/2] - 1$ *connecting lines*. If true, then this would be best possible, as shown by the configuration of n points with no point lying on more than $[n/2] - 1$ connecting lines (Dirac 1951; Motzkin 1975).

Graham conjectured that any subset of P which meets all the connecting lines contains all the points on at least one connecting line. This conjecture was proved by M. Rabin and independently by Motzkin (see Erdös 1975, pp. 105–106).

Erdös asked: If $t_i(P) = 0$ for $i \geqslant 4$, must P necessarily contain three points, say p_1, p_2, p_3, such that all three connecting lines p_1p_2, p_2p_3, p_3p_1 are ordinary? A recently constructed example of Füredi and Palásti shows that the answer is NO (Erdös 1983).

A set of n points which spans 3-space determines at least n connecting planes (Motzkin 1951). Hanani (1954) investigated the (three) exceptional cases, where the n points determine exactly n planes. Basterfield and Kelly (1968) gave conditions under which a set of n points in d-dimensional space determine precisely n hyperplanes.

5. Finite collections of sets

In this section Λ will denote a finite collection of disjoint sets whose union spans the plane. Must there exist a line intersecting precisely two members of Λ? If the sets are singletons this is just Sylvester's problem. We shall examine conditions which must be imposed on Λ to guarantee the existence of such a line. We call a line Λ-ordinary if it meets exactly two of the sets in Λ.

Figure 10 (Edelstein and Kelly 1966; Grünbaum 1975) exhibits a collection $\Lambda = \{A, B, C\}$ with $|A| = |B| = |C| = 4$ determining no Λ-ordinary line. This example,

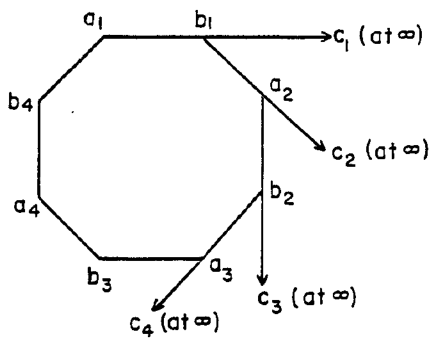

Figure 10.

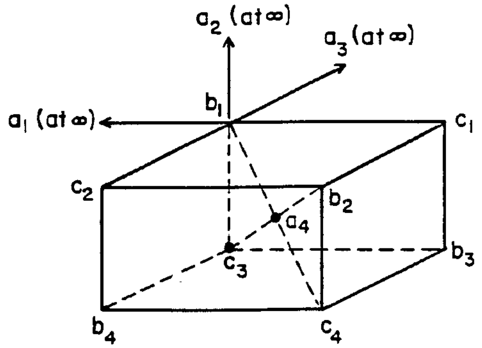

Figure 11.

which can be extended to any regular $2n$-gon, shows that requiring the members of Λ to be finite does not guarantee the existence of a Λ-ordinary line. These configurations are the only known class of counter-examples to the generalized problem for collections of finite sets on the plane. It is tempting to speculate that there cannot be too many other configurations of this sort.

In three dimensions the Desmic configuration (Figure 11) also has sets A, B, C with $|A| = |B| = |C| = 4$ and no ordinary Λ-line. Edelstein and Kelly (1966) conjectured that it is the unique configuration Λ of finite sets in 3-space that contains no ordinary Λ-line. This startling conjecture was proved for collections of three finite sets by Borwein (1983b).

Edelstein and Kelly (1966) observed that any collection Λ of finite sets in E^k ($k \geqslant 4$) determines an ordinary line, for otherwise a suitable projection onto a plane would violate Sylvester's theorem.

Consider three compact and countable sets A, B and C in E^2 that are not on a line. Assume one of the sets, say A, is infinite and that every line through two of the sets intersects the third set. We offer a simple proof that this is not possible. Let a be a limit point of A and consider the projection π of $B \cup C$ from a to any line not passing through a, as in Figure 12. Let b be any point of B and assume that the line joining b to a does not contain all but finitely many of the points of A (there can be at most one such line). There is a sequence $\{a_n\}$ tending to a, and by assumption there is a point c on the line $l(a, b)$, joining a and b. Consider the lines joining each of the a_i to c. Each of these lines contains a point b_i and one verifies that the sequence $\{\pi(b_i)\}$ tends to $\pi(b)$. In particular, every point of $\pi(B \cup C)$ (except possibly one point) is a limit point of $\pi(B \cup C)$. This, however, is impossible since $\pi(B \cup C)$ is countable.

We examine the situation where at least one member of $\Lambda = \{A_1, A_2, \ldots, A_r\}$ is infinite. The first result, due to Grünbaum (1956) asserts that all connected and

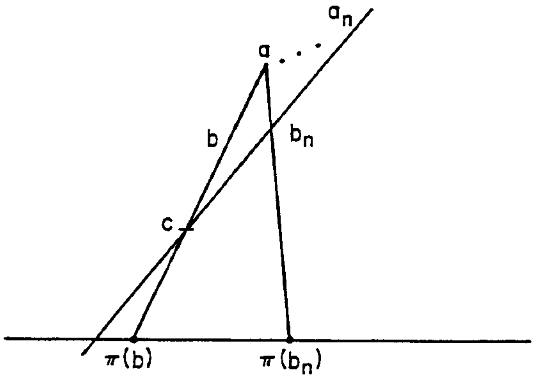

Figure 12.

compact A_i in the plane contain an $\{A_i\}$-ordinary line. This was extended by Edelstein (1957) to cover the case where each A_i consists of finitely many compact connected components, and then Herzog and Kelly (1960) showed that the connectedness hypothesis can be dropped.

Edelstein, Herzog and Kelly (1963) show that in E^n a finite collection of disjoint compact sets (at least one of the sets infinite) generates a hyperplane cutting exactly two of the sets. Edelstein and Kelly (1966) relocated the problem in real normed linear spaces and extended the results to this setting. (See also Edelstein 1969.) They prove the following: Let $\Lambda = \{S_i\}$ be a finite collection of disjoint compact sets in a real normed linear space. Then one of the following must hold:

(a) there exists a hyperplane intersecting exactly two of the sets;
(b) $\bigcup S_i$ lies on a line;
(c) $\bigcup S_i$ is finite and spans a space of dimension 2 or 3.

In most situations, instead of considering disjoint and compact A_i, it suffices to consider sets A_i whose closures are compact and disjoint. The property that every line through two sets intersects a third set carries through in the limit, provided that the closures are disjoint.

6. Colored pointsets and monochromatic linear subsets

Let $\{P_i\}$ be a collection of sets of points. Call a line *monochrome* if it passes through at least two points of one of the P_i and does not intersect any of the others. More generally, any geometric structure (hyperplane, vertex, etc.) will be called monochrome if it is defined by and is incident with exactly one of the sets.

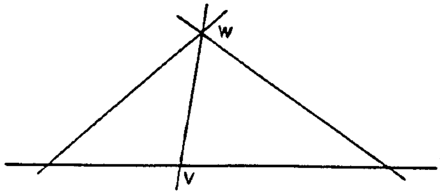

Figure 13.

R. Graham asked the following question: If two finite disjoint sets of points span the plane must there exist a monochrome line? The affirmative answer was provided by T. Motzkin (1967). (See also Edmonds, Lovász and Mandel 1980.) We outline his simple and elegant argument for the dual formulation of the problem. Suppose there exist two sets S and T of nonconcurrent lines that define no monochrome vertices. Consider the smallest (in area) configuration of this type, presented in Figure 13, that is a triangle with edges from both sets and with the mandatory extra S-line at w (the vertex formed by two edges from the same set, say T) cutting the interior of the triangle. An initial collineation, if necessary, ensures that such a configuration exists. Since vertex v is not monochrome an additional s-line passes through it and we have generated a smaller configuration of the prescribed variety. This proof can easily be made projective by considering contained figures.

Chakerian (1970) offers an entirely different proof of the answer to Graham's question based on a combinatorial lemma of Cauchy that counts sign changes at vertices of maps on the sphere. Another proof may be found in P. Borwein (1982).

Shannon (1974) proves that n finite and disjoint sets whose union spans real n-space (Euclidean, affine or projective) suffice to guarantee the existence of a monochrome line. The proof, once again in the dual formulation, is a generalization of Motzkin's argument. In 3-space, for example, consider a minimal tetrahedron formed by planes from all three sets with the mandatory additional plane cutting the interior of the tetrahedron, and proceed as before.

A direct proof of the following theorem which inductively subsumes the previous results may be found in P. Borwein (1982). Given two finite and disjoint sets A and B whose union spans real n-space then either there exists a monochrome line spanned by points of A or there exists a monochrome hyperplane spanned by points of B.

A conjecture of P. Borwein and M. Edelstein (1983) that would cover all these cases is: Given two disjoint finite sets A and B whose union spans real $(n+m)$-space then either there exists an A-monochrome n-flat or there exists a B-monochrome m-flat, where by an A-monochrome n-flat we mean an affine variety of dimension n spanned by points of A that contains no points of B.

The first unresolved case concerns monochrome planes in four dimensions. (See Baston and Bostock (1978) for related results.)

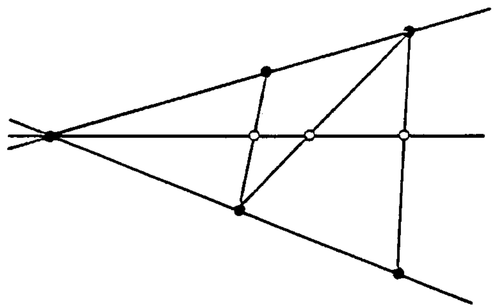

Figure 14.

Figure 14 shows that in the plane it is not possible to combine Motzkin's result and Sylvester's theorem. We cannot guarantee the existence of an ordinary monochrome line.

Grünbaum (1975) (see also Shannon (1974)) conjectures that: Given n A-lines and m B-lines in the plane, if there exist no monochrome A-vertices and the B-lines are not all concurrent, then $n - m \leqslant 4$. Otherwise little attention seems to have been paid to questions concerning related combinatorial questions.

As with Sylvester's problem, the monochrome line problem has been extended to different sets. Tingley (1975) shows that disjoint compact connected sets whose union spans the plane define a monochrome line. J. Borwein (1979) shows that it suffices that the two sets have disjoint connected closures and that one of them be bounded. In three dimensions Tingley (1975) proves that the connectedness hypothesis is no longer necessary; in fact, two sets with disjoint compact closures whose union spans E^3 generate a monochrome line.

In two dimensions this, as Tingley points out, is false (See Figure 15).

At the other end of the connectedness spectrum Tingley (1975) and J. Borwein (1980) ask whether Graham's original question extends to countable and compact

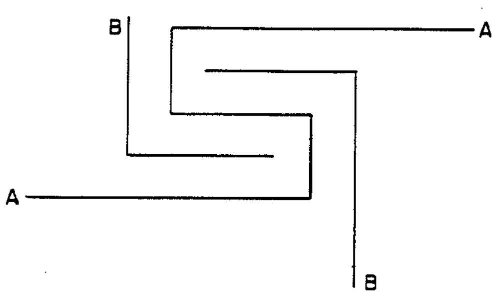

Figure 15.

sets in the plane. This is established (P. Borwein 1984) using topological arguments. Unfortunately the proof does not also cover the finite case.

Most of these results can be formulated in higher dimensions. One attractive question is: Do k countable, compact, disjoint sets in E^k insure the existence of a monochrome line?

Baston and Bostock (1978) call two disjoint sets n-incident if every flat spanned by n distinct points of one set contains a point of the other. This is of a slightly different flavor from previous concerns since, for example, two 3-incident sets in space contain no monochrome planes and also contain no monochrome lines through three points. If $M(n)$ denotes the least integer so that two finite n-incident sets necessarily lie in an $M(n)$-flat then they show that

$$2n-3 \leqslant M(n) \leqslant 4n-6$$

and conjecture that

$$M(n) = 2n-3.$$

Note that the previously mentioned conjecture of Borwein and Edelstein (1983) would imply this.

REFERENCES

MR = Math. Rev. Zbl = Zentralblatt FM = Fortschritte der Math.

BALOMENOS, R. H., BONNICE, W. E. and SILVERMAN, R. J. 1966. *Extensions of Sylvester's theorem.* Canad. Math. Bull. *9,* 1–14. MR: *33,* #6477.

BASTERFIELD, J. G. and KELLY, L. M. 1968. *A characterization of sets of n points which determine n hyperplanes.* Proc. Cambridge Philos. Soc. *64,* 585–588. MR: *38,* #2040.

BASTON, V. J. and BOSTOCK, F. A. 1978. *A Gallai-type problem.* J. Combin. Theory Ser. A *24,* 122–125. MR: *57,* #150.

BECK, J. 1983. *On a "lattice property" of the plane and some problems in combinatorial geometry.* Combinatorica *3,* 281–297. Zbl 533.52004.

BONNICE, W. and EDELSTEIN, M. 1967. *Flats associated with finite sets in P^d.* Niew Arch. Wisk. *15,* 11–14. MR: *35,* #7189.

BONNICE, W. and KELLY, L. M. 1971. *On the number of ordinary planes.* J. Combin. Theory *11,* 45–53. MR: *45,* #1784.

BOROS, E., FÜREDI Z. and KELLY, L. M. *On representing Sylvester-Gallai designs.* Discrete Comput. Geom. *4,* 345–348.

BORWEIN, J. 1979. *Monochrome lines in the plane.* Math. Mag. *52,* 41–45.

BORWEIN, J. 1980. *Problems 297, 298.* Canad. Math. Bull. *23,* 506.

BORWEIN, P. 1982. *On monochrome lines and hyperplanes.* J. Combin. Theory Ser. A *33,* 76–81.

BORWEIN, P. 1983a. *On Sylvester's problem and Haar spaces.* Pacific J. Math. *109,* 275–278.

BORWEIN, P. 1983b. *The Desmic conjecture.* J. Combin. Theory Ser. A *35,* 1–9.

BORWEIN, P. 1984. *Sylvester's problem and Motzkin's theorem for countable and compact sets.* Proc. Amer. Math. Soc. *90,* 580–584.

BORWEIN, P. and EDELSTEIN, M. 1983. *A conjecture related to Sylvester's problem*. Amer. Math. Monthly *90*, 389–390.

BRAKKE, K. A. 1972. *Some new values of Sylvester's function for n noncollinear points*. J. Undergrad. Math. *4*, 11–14.

BRUIJN, N. G. DE and ERDŐS, P. 1948. *On a combinatorial problem*. Neder. Akad. Wetensch. Proc. *51*, 1277–1279 (=Indagationes Math *10*, 421–423). MR: *10*, p. 424.

BURCKHARDT, J. J. 1940. *Über konvexe Körper mit Mittlepunkt*. Vierteljahrsschr. Naturforsch: Ges. Zürich *85*, 149–154.

BURR, S. A., GRÜNBAUM, B. and SLOANE. N. J. A. 1974. *The Orchard Problem*. Geom. Dedicata *2*, 397–424. MR: *49*, #2428.

CHAKERIAN, G. D. 1970. *Sylvester's problem on collinear points and a relative*. Amer. Math. Monthly *77*, 164–167. MR: *41*, #3305.

COXETER, H. S. M. 1984. *A problem of collinear points*. Amer. Math. Monthly *55*, 26–28. MR: *9*, p. 458.

COXETER, H. S. M. 1955. *The Real Projective Plane* (2nd ed.). Cambridge University Press, Cambridge. MR: *10*, p. 729.

COXETER, H. S. M. 1968. *Twelve Geometric Essays*. Southern Illinois University Press, Carbondale, IL. MR: *46*, #9843.

COXETER, H. S. M. 1969. *Introduction to Geometry* (2nd ed.). Wiley, New York. MR: *49*, #11369.

COXETER, H. S. M. 1973. *Regular Polytopes* (3rd ed.). Dover, New York. MR: *51*, #6554.

CROFT, H. T. 1967. *Incidence incidents*. Eureka (Cambridge) *30*, 22–26.

CROWE, D. W. and MCKEE, T. A. 1968. *Sylvester's problem on collinear points*. Math. Mag. *41*, 30–34. MR: *38*, #3761.

DIRAC, G. A. 1951. *Collinearity properties of sets of points*. Quart. J. Math. *2*, 221–227. MR: *13*, p. 270.

DIRAC, G. A. 1959. *Review of Kelly and Moser (1958)*. MR: *20*, #3494.

EDELSTEIN, M. 1957. *A further generalization of a problem of Sylvester*. Riveon Lamatematika *11*, 50–55. Hebrew. English summary. MR: *21*, #2211.

EDELSTEIN, M. 1969. *Hyperplanes and lines associated with families of compact sets in locally convex spaces*. Math. Scand. *25*, 25–30. MR: *41*, #983.

EDELSTEIN, M., HERZOG, F. and KELLY, L. M. 1963. *A further theorem of the Sylvester type*. Proc. Amer. Math. Soc. *14*, 359–363. MR: *26*, #4333.

EDELSTEIN, M. and KELLY, L. M. 1966. *Bisecants of finite collections of sets in linear spaces*. Canad. J. Math. *18*, 375–380. MR: *32*, #6185.

EDMONDS, J., LOVÁSZ, and MANDEL, A. 1980. *Solution to problem in vol. 1, p. 250*. Math. Intelligencer *2*, 106–107.

ERDŐS, P. 1943. *Problem 4065*. Amer. Math. Monthly *50*, 65.

ERDŐS, P. 1944. *Solution of Problem 4065*. Amer. Math. Monthly *51*, 169–171.

ERDŐS, P. 1972. *On a problem of Grünbaum*. Canad. Math. Bull. *15*, 23–25. MR: *47*, #5709.

ERDŐS, P. 1973. *Problems and results in combinatorial analysis*. (Technion Preprint Series No. MT-182). Technion, Haifa.

ERDŐS, P. 1975. *On some problems of elementary and combinatorial geometry*. Ann. Mat. Pura Appl. (4) *103*, 99–108. MR: *54*, #113.

ERDŐS, P. 1980. *Some combinatorial problems in geometry*. Geometry and Differential Geometry (Proc. Conf. Univ. Haifa 1979), 46–53. (Lecture Notes in Math., Vol. 792). Springer, Berlin.

ERDŐS, P. 1982. *Personal reminiscences and remarks on the mathematical work of Tibor Gallai*. Combinatorica *2*, 207–212.

ERDŐS, P. 1983. *Combinatorial problems in geometry*. Math. Chronicle *12*, 35–54.

ERDŐS, P. 1984a. *Some old and new problems on combinatorial geometry*. Convexity and Graph Theory (Jerusalem, 1981), 129–136. (North-Holland Math. Studies, Vol. 87). MR: *87b*, #52018.

ERDŐS, P. 1984b. *Some old and new problems on combinatorial geometry*. Annals Discrete Math. *20*, 129–136.

ERDŐS, P. 1984c. *Research problems*. Period Math. Hung. *15*, 101–103, Zbl 537.0515.

ERDÖS, P. 1985. *Problems and results in combinatorial geometry*. Discrete geometry and convexity (New York, 1982), 1–11. (Ann. New York Acad. Sci., Vol. 440). MR: *87g*, #52001.

ERDÖS, P. and PURDY, G. 1976. *Some extremal problems in geometry*. IV. Proceedings of the Seventh Southeastern Conference on Combinatorics, Graph Theory and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1976, 307–322. (Congressus Numerantium, Vol. XVII). Utilitas Math., Winnipeg, Man. MR: *55*, #10292.

GALLAI, T. 1944. *Solution to Problem 4065*. Amer. Math. Monthly *51*, 169–171.

GRÜNBAUM, B. 1956. *A generalization of a problem of Sylvester*. Riveon Lematematika *10*, 46–48. (Hebrew. English summary) MR: *19*, p. 667.

GRÜNBAUM, B. 1972. *Arrangements and Spreads*. (CBMS, No. 10). Amer. Math. Soc., Providence, RI. MR: *46*, #6148.

GRÜNBAUM, B. 1975. *Arrangements of colored lines*. Notices Amer. Math. Soc. *22*, A-200.

GRÜNBAUM, B. 1976. *New views on some old questions of combinatorial geometry*. (Colloq. Int. Theorie Comb., Roma 1973, Tomo 1), 451–468. MR: *57*, #10605. Zbl 347.52004.

GRÜNBAUM, B. and SHEPHARD, G. C. 1984. *Simplicial arrangements in projective 3-space*. (Coxeter Festschrift) Mitt. Math. Sem. Giessen *166*, 49–101.

GUY, R. K. 1971. *Problem 15*. Unsolved combinatorial problems. Combinatorial Mathematics and its Applications (Proceedings Conf. Oxford 1969, Editor, D. J. A. Welsh), 121–127. Academic Press, London. MR: *43*, #3126.

HADWIGER, H., DEBRUNNER, H. and KLEE, V. 1964. *Combinatorial Geometry in the Plane*. Holt, Rinehart and Winston, New York. MR: *29*, #1577.

HANANI, H. 1951. *On the number of straight lines determined by n points*. Riveon Lematematika *5*, 10–11. MR: *13*, p. 5.

HANANI, H. 1954/5. *On the number of lines and planes determined by d points*. Technion Israel Tech. Sci. Publ. *6*, 58–63.

HANSEN, S. 1965. *A generalization of a theorem of Sylvester on the lines determined by a finite point set*. Math. Scand. *16*, 175–180. MR: *34*, #3411.

HANSEN, S. 1980. *On configurations in 3-space without elementary planes and on the number of ordinary planes*. Math. Scand. *47*, 181–194. Zbl 438.51002.

HANSEN, S. 1981. *Contributions to the Sylvester–Gallai theory*. Ph.D. Thesis. Copenhagen.

HERZOG, F. and KELLY, L. M. 1960. *A generalization of the theorem of Sylvester*. Proc. Amer. Math. Soc. *11*, 327–331. MR: *22*, #4044.

KÁRTESZI, F. 1963. *Alcuni problemi della geometria d'incindenza*. (Conf. Sem. Math. Univ. Bari, No. 88). FM: *31*, #3926.

KELLY, L. M. 1986. *A resolution of the Sylvester–Gallai problem of J.-P. Serre*. Discrete Comput. Geom. *1*, 101–104.

KELLY, L. M. and MOSER, W. 1958. *On the number of ordinary lines determined by n points*. Canad. J. Math. *10*, 210–219. MR: *20*, #3494.

KELLY, L. M. and ROTTENBERG, R. 1972. *Simple points on pseudoline arrangements*. Pacific J. Math. *40*, 617–622. MR: *46*, #6150.

LANG, G. D. W. 1955. *The dual of a well-known theorem*. Math. Gazette *39*, 314.

MELCHIOR, E. 1940. *Über Vielseite der projektiven Ebene*. Deutsche Math. *5*, 461–475. MR: *3*, p. 13.

MEYER, W. 1974. *On ordinary points in arrangements*. Israel J. Math. *17*, 124–135. MR: *52*, #148.

MOSER, W. 1957. *Abstract Groups and Geometrical Configurations*. Ph.D. Thesis. Univ. of Toronto.

MOSER, W. 1961. *An enuneration of the five parallelohedra*. Cand. Math. Bull. *4*, 45–47. MR *24a*, #3556.

MOTZKIN, T. 1951. *The lines and planes connecting the points of a finite set*. Trans. Amer. Math. Soc. *70*, 451–464. MR: *12*, p. 849.

MOTZKIN, T. 1967. *Nonmixed connecting lines*. Abstract 67T-605. Notices Amer. Math. Soc., *14*, 837.

MOTZKIN, T. 1975. *Sets for which no point lies on many connecting lines*. J. Combinatorial Theory *A18*, 345–348. MR *51*, #2946.

ROTTENBERG, R. 1971. *On finite sets of points in P^3*. Israel J. Math. *10*, 160–171. MR: *45*, #5879.

ROTTENBERG, R. 1973. *Exceptional sets in a projective plane*. (Technion Preprint Series, No. MT-169). Technion, Haifa.

ROTTENBERG, R. 1981. *Visibility and parallelotopes in projective space*. J. Geometry *16*, 19–29. MR: *83b*, 51005. Zbl 473.51012.

ROTTENBERG, R. 1982. *Singular points in planar finite sets I.* Geometriae Dedicata *12*, 407–415. Zbl 492.05021.

SAH, CHIH-HAN. 1986. *The rich line problem of P. Erdös*. (Colloquia Math. Soc. János Bolyai, Vol. 48. Intuitive Geometry, Siófok, 1985), 123–125.

SALAMON, P. and ERDÖS, P. 1988. *The solution to a problem of Grünbaum*. Canad. Math. Bull. *31*, 129–138.

SHANNON, R. 1974. Ph.D. Thesis. Univ. Washington, Seattle.

STEINBERG, R. 1944. *Three point collinearity*. Amer. Math. Monthly *51*, 169–171.

SYLVESTER, J. J. 1893. *Mathematical Question 11851*. Educational Times *59*, 98.

SZEMERÉDI, E. and TROTTER Jr., W. T. 1983a. *A combinatorial distinction between the Euclidean and projective plane*. European J. Combin. *4*, 385–394. Zbl 539.05026.

SZEMERÉDI, E. and TROTTER Jr., W. T. 1983b. *Recent progress on extremal problems in discrete geometry*. (Graphs and other combinatorial topics, Prague 1982). (Teubner-Text zur Math. Vol. 59), 316–319. Teubner, Leipzig.

SZEMERÉDI, E. and TROTTER Jr., W. T. 1983c. *Extremal problems in discrete geometry*. Combinatorica *3*, 381–392. Zbl 541.05012.

TINGLEY, D. 1975. *Monochromatic lines in the plane*. Math. Mag. *48*, 271–274, MR: *53*, #9115.

WILLIAMS, V. C. 1968. *A proof of Sylvester's theorem on collinear points*. Amer. Math Monthly. *75*, 980–982.

WATSON, K. S. 1981. *Sylvester's problem for spreads of curves*. Canad. J. Math. *32*, 219–239, Zbl 436. 51010.

WOODALL, D. R. 1969. *The λ-μ problem*. J. Lond. Math. Soc. (2) *1*, 509–519.

Department of Mathematics,
McGill University,
Burnside Hall,
805 Sherbrooke St. West,
Montreal, Quebec, Canada.
H3A 2K6